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Math 3150 Problems
Haberman Chapter H2

Due Date: Problems are collected on Wednesday.

Chapter H2: 2.5 Laplace's Equation, Maximum Principle

Problem H2.5-1. (Laplace's Equation on a Rectangle, Temperature and Insulation Conditions)

Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < L$, $0 < y < H$ subject to the following boundary conditions.

(a) $u_x(x,y) = 0$ for $x = 0$ and $x = L$, $u(x,y) = 0$ for $y = 0$, $u(x,y) = f(x)$ for $y = H$

(b) $u_x(x,y) = 0$ for $x = 0$, $u(x,y) = g(y)$ for $x = L$, $u(x,y) = 0$ for $y = 0$ and $y = H$

(c) $u(x,y) = 0$ for $x = 0$ and $x = L$, $u(x,y) - u_y(x,y) = 0$ for $y = 0$, $u(x,y) = f(x)$ for $y = H$

Reference. Haberman H2.5. See Exercises 1(a), 1(c), 1(e), which have answers.

Problem H2.5-2. (Laplace's Equation on a Rectangle, Insulated Boundary)

Consider $u(x,y)$ satisfying Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < L$, $0 < y < H$ subject to the insulated boundary conditions

$$u_x(0,y) = u_x(L,y) = 0,\ \ u_y(x,0) = 0,\ \ u_y(x,H) = f(x).$$

(a) The conditions describe the heat flux normal to the boundary. For example, along boundary segment $x = 0$ the outer normal is $\vec{n} = -\vec{\imath} + 0\vec{\jmath}$ and the normal component of the temperature gradient $\mathbf{grad}(u) = u_x\vec{\imath} + u_y\vec{\jmath}$ is $\mathbf{grad}(u)\cdot\vec{n} = -u_x$. The net heat flow along the segment is then $\int_0^H -u_x(0,y)dy$. Compute the net heat flow, summed across all four boundary segments.

(b) The temperature $U(x,y,t)$ at equilibrium $t = \infty$ becomes $u(x,y)$, independent of time. The net heat flow of $u(x,y)$ across the rectangle boundary must be zero. Otherwise, there is heat transfer and temperature change in time. Apply part (a) to write net heat flow zero as a condition for solvability of the problem.

(c) Assume the extra condition found in part (b). Solve for $u(x,y)$ by the method of separation of variables.

Remark. *See (61) in H2.5, for zero net heat flow. The solution will contain an unresolved constant C, which is determined by the time-dependent initial condition $U(x,y,0) = g(x,y)$. Don't bother to find C.*

Problem H2.5-3. (Laplace's Equation Outside a Disk)

Solve Laplace's equation outside a circular disk ($r > a$) subject to the boundary condition

(a) $u(a,\theta) = \ln(2) + 4\cos(3\theta)$

(b) $u(a,\theta) = f(\theta)$

References. Assume that $u(r,\theta)$ remains finite at r=infinity to obtain Haberman's answer

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta))r^{-n}.$$

Part (a) is fast because $u(a,\theta)$ is a linear combination of eigenfunctions.

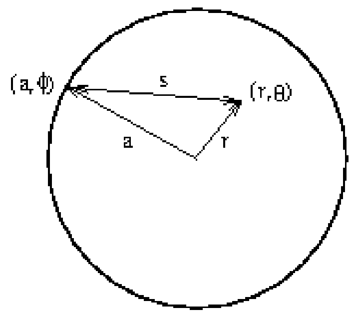
Part (b) uses orthogonality of the eigenfunctions to write Fourier coefficient formulas for A_0, A_n, B_n in terms of $f(\theta)$.

Problem H2.5-4. **(Poisson's Integral Formula)**

For Laplace's equation inside a circular disk ($r < a$), the series solution formula can be re-arranged into Poisson's integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)K(r,\theta,\phi)d\phi, \quad \text{where}$$
$$K(r,\theta,\phi) = \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2} = \text{ Poisson's Kernel.}$$

(a) In the figure, s is the distance from (r,θ) to (a,ϕ). Derive $s^2 = a^2 - 2ar\cos(\theta-\phi)+r^2$ from the law of cosines and polar coordinates, then conclude that the Poisson Kernel is the formula $K = \dfrac{a^2-r^2}{s^2}$. This also justifies $K > 0$.



(b) Use the complex exponential identity $\cos(w) = \dfrac{e^{iw}+e^{-iw}}{2}$ and the geometric series formula $\sum_{j=0}^{\infty} z^j = \dfrac{1}{1-z}$ to derive the identity

$$K(r,\theta,\phi) = -1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi)).$$

(c) Use $f(\theta) = 1$ and solution $u(r,\theta) = 1$ to derive from Poisson's formula the identity

$$\frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,\phi)d\phi = 1.$$

The identity can also be derived by integrating the part (b) formula, because $\int_0^{2\pi}\cos(n(\theta-\phi))d\phi = 0$.

(d) Poisson's formula says that $u(r,\theta)$ is a weighted average of the boundary data $f(\theta)$ on the circle, with weight function K. Understand this by showing that $u(r,\theta)$ at $r = 0$ is the average value of f on the circle (Mean Value Theorem), using $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} Kf\,d\phi$.

Problem H2.5-5a. **(Laplace's Equation on a Quarter Circle)**

Solve Laplace's equation inside the quarter-circle of radius 1 , $0 < \theta < \pi/2$, $0 < r < 1$, subject to the boundary conditions $\frac{\partial u}{\partial\theta}(r,0) = 0$, $u(r,\pi/2) = 0$, $u(1,0) = f(\theta)$.

Problem H2.5-5b. **(Laplace's Equation on a Disk: Graphics)**

Laplace's equation $u_{xx} + u_{yy} = 0$ on a disk of radius 1 is to be solved with initial data $u(1,\theta) = \cos(10\theta)$, $|\theta| < \pi$. The solution is $u(r,\theta) = r^{10}\cos(10\theta)$.

(a) Explain why the answer is so simple, and how it was obtained.

(b) Make a technology plot of the solution, which is temperature u in the z-axis direction, with x, y restricted to the disk $x^2 + y^2 \le 1$. Clip the negative temperatures. Show good detail near the edge of the disk, because r^{10} makes $u = 0$ near $r = 0$ (the $z = 0$ plane) and most of the way out to the edge of the disk. Color plots which can be shown on your laptop are appreciated.

Problem XC-H2.5-8a. **(Laplace's Equation inside a Circular Annulus)**

Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions $u(a,\theta) = f(\theta)$, $u(b,\theta) = g(\theta)$.