Partial Differential Equations 3150

Sample Final Exam

Exam Date: Friday, April 25, 2014

Instructions: This exam is timed for 120 minutes. You may be given extra time to complete the exam (10-15 extra minutes). No calculators, notes, tables or books. Problems use topics from the required textbook which were covered in lectures. Details count 3/4, answers count 1/4.

1. (Chapters H1-H2. Heat Conduction in a Bar)

Considered is the heat conduction problem

$$\begin{cases} u_t & = & \frac{1}{4}u_{xx}, & 0 < x < 10, & t > 0, \\ u(0,t) & = & 50, & & t > 0, \\ u(10,t) & = & 80, & & t > 0, \\ u(x,0) & = & f(x), & 0 < x < 10. & \end{cases} \quad (1)$$

It represents a laterally insulated uniform bar of length 10 with one end at 50 Celsius and the other end at 80 Celsius, and initial temperature $f(x)$.

1(a) [30%] Show the details for finding the the steady-state temperature $u_1(x) = 50 + 3x$.

1(b) [40%] Show the details for the solution $u_2(x,t) = \sin(\pi x/10)\, e^{-\pi^2 t/400}$ of the ice-pack ends bar problem

$$\begin{cases} u_t & = & \frac{1}{4}u_{xx}, & 0 < x < 10, & t > 0, \\ u(0,t) & = & 0, & & t > 0, \\ u(10,t) & = & 0, & & t > 0, \\ u(x,0) & = & \sin(\pi x/10), & 0 < x < 10. & \end{cases}$$

1(c) [30%] Superposition implies $u(x,t) = u_1(x,t) + u_2(x,t) = 50 + 3x + \sin(\pi x/10)\, e^{-\pi^2 t/400}$ is a solution of (1) with $f(x) = -50 - 3x + \sin(\pi x/10)$. Show the details of an answer check for this solution $u(x,t)$.

1(a) $u_{xx} = 0$, $u = c_1 + c_2 x$; $u(0) = 50$ and $u(10) = 80 \Rightarrow c_1 = 50, c_2 = 3$.

1(b) Product solutions $u = X(x)T(t)$, $X'' + \lambda X = 0$, $X(0) = X(10) = 0$, $T' + \frac{1}{4}\lambda T = 0$, $T \neq 0$. Then $X = \sin(\sqrt{\lambda}x)$ with $\lambda > 0$ and $10\sqrt{\lambda} = n\pi$. Finally, $u =$ superposition $= \sum_1^\infty a_n \sin\left(\frac{n\pi}{10}x\right) e^{-n^2\pi^2 t/400}$ because $T = e^{-\frac{1}{4}\lambda t}$. Because $u(x,0) =$ eigenfunction for $\lambda = 1 \cdot \pi/10$, then $u(x,t) = a_1 \sin(\pi x/10)\, e^{-\pi^2 t/400}$ with $a_1 = 1$.

1(c) $u_t = \frac{\partial}{\partial t}u_1 + \frac{\partial}{\partial t}u_2 = 0 + \frac{1}{4}\frac{\partial^2}{\partial x^2}u_2 \left(= -\frac{\pi^2}{400}\sin\left(\frac{\pi x}{10}\right)e^{-\frac{\pi^2 t}{400}}\right)$ but not needed

$\frac{1}{4}u_{xx} = \frac{1}{4}\frac{\partial^2}{\partial x^2}u_1 + \frac{1}{4}\frac{\partial^2}{\partial x^2}u_2 = 0 + \frac{1}{4}\frac{\partial^2}{\partial x^2}u_2 = u_t$

So the PDE is satisfied. Then $u(0,t) = u(10,t) = 0$ because $\sin(0) = 0$. And finally,

$$u(x,0) = \sin(\pi x/10)e^0 = \sin\left(\frac{\pi x}{10}\right).$$

Done.

Use this page to start your solution. Attach any extra pages, then re-staple.

2. (Chapter H3. Fourier Series)

2(a) [20%] Find and display the nonzero terms in the Fourier series expansion of $f(x)$, formed as the even 2π-periodic extension of the function $f_0(x) = \sin^2(x) + 4\cos(2x)$ on $0 < x < \pi$.

2(b) [40%] Let $g(x)$ be the Fourier sine series for the the period 2 odd extension of the function $g_0(x) = 1$ on $0 \le x \le 1$. Complete the following.

(1) Graph of g_0 with its odd extension on $|x| < 1$.
(2) Graph of $g(x)$ over 4 periods.
(3) Fourier sine series coefficient formulas.
(4) Numerical values for the coefficients.
(5) Gibb's overshoot graphic on $|x| < 2$.

2(c) [20%] Define $h_0(x) = \begin{cases} \sin(2x) & 0 \le x < \pi, \\ x - \pi & \pi \le x \le 2\pi, \end{cases}$ and let $h(x)$ be the 4π odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $h(-5.25\pi) + h(1.5\pi)$.

2(d) [20%] Compute the smallest period, for those functions which are periodic.

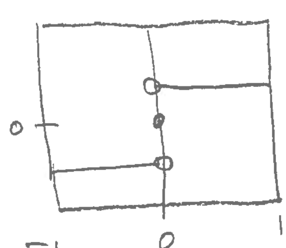
(1) $f_1(x) = 2\sin(x) + 5\cos(x)$ period $= 2\pi$
(2) $f_2(x) = \sin(x) + \cos(5x)$ period $= 2\pi$
(3) $f_3(x) = \sin(x) + \cos(\pi x)$ Not periodic
(4) $f_4(x) = \sin(x) + \sin(x)\cos(x) = \sin x + \frac{1}{2}\sin 2x$ period $= 2\pi$

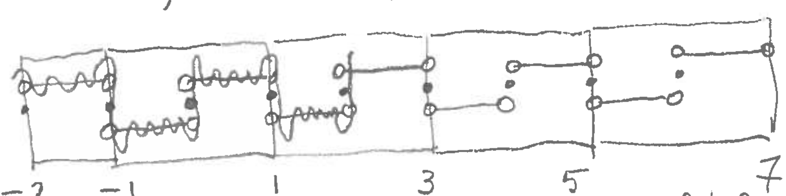
2(a) $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ [used $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\cos^2\theta + \sin^2\theta = 1$]

$f_0(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) + 4\cos(2x)$ is even

$f(x) = f_0(x)$ on $|x| < \pi$, $f(x) = \frac{1}{2} + \frac{7}{2}\cos(2x)$ is The Fourier Series

2(b)





There are other possibilities, for f, but not for The Fourier Series

One extension

$$\frac{f(x+)+f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right), \quad b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$b_n = 2\int_0^1 \sin(n\pi x)\,dx = 2\left(\frac{1-\cos(n\pi)}{n\pi}\right).$$ Overshoot shown on -2 to 2.

2(c) h odd of period 4π, so $h(-5.25\pi) = h(-5.25\pi + 4\pi) = h(-1.25\pi)$
$= -h(1.25\pi)$ because h is odd. Then $h(-5.25\pi) + h(1.5\pi) =$
$-h(1.25\pi) + h(1.5\pi) = -(1.25\pi - \pi) + (1.5\pi - \pi) = 0.25\pi = \boxed{\frac{\pi}{4}}$

2(d) 2π, 2π, Not periodic, 2π. Some details, next page.

→ Show $f_3(x) = \sin x + \cos \pi x$ is not periodic.

Suppose $f_3(x+p) = f_3(x)$ for some $p > 0$. we use

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$f_3(x+p) = f_3(x)$$

Expand ↓

$$\sin(x+p) + \cos(\pi x + \pi p) = \sin(x) + \cos(\pi x)$$

use identities

$$\cos(p)\sin(x) + \sin(p)\cos(x) + \cos(\pi p)\cos(\pi x) - \sin(\pi p)\sin(\pi x)$$
$$= \sin(x) + \cos(\pi x).$$

By independence of the functions $\sin x$, $\cos x$, $\cos \pi x$, $\sin \pi x$ the coefficients left and right must match:

$$\begin{cases} \cos(p) = 1 \\ \sin(p) = 0 \\ \cos(\pi p) = 1 \\ \sin(\pi p) = 0 \end{cases}$$

From the first two, $p = n\pi$ with n even and positive

From the last, $\pi p = m\pi$ for $m > 0$. Then $p = m$, giving

$$m = p = n\pi$$

or

$$\pi = \frac{m}{n}$$

But π is not a rational number. So f_3 cannot be periodic of any period.

3. (CH H4. Finite String: Fourier Series Solution)

3(a) [50%] Complete the following for the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0 < x < 2, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(2,t) &= 0, & & t > 0, \\ u(x,0) &= f(x), & 0 < x < 2, & \\ u_t(x,0) &= g(x), & 0 < x < 2. & \end{cases}$$

(1) Separation of variables details.
(2) Product solutions and boundary conditions.
(3) The normal modes.
(4) Superposition.
(5) Series solution $u(x,t)$.

3(b) [25%] Display explicit formulas for the generalized Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

3(c) [25%] Evaluate the coefficients when $f(x) = 100$ and $g(x) = 0$.

3(a) $u = X(x)T(t)$, $\dfrac{X''}{X} = \dfrac{T''}{\frac{1}{4}T} = -\lambda$, $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(2) = 0 \end{cases}$ and $\begin{cases} T'' + \frac{1}{4}\lambda T = 0 \\ T \neq 0 \end{cases}$

Then $\lambda > 0$, $2\sqrt{\lambda} = n\pi$, $X = \sin\left(\frac{n\pi x}{2}\right)$, $T = a_n \cos\left(\frac{n\pi}{2} \cdot \frac{1}{2} t\right) + b_n \sin\left(\frac{n\pi}{2} \cdot \frac{1}{2} t\right)$

Normal modes $=$ $\sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi t}{4}\right)$, $\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{n\pi t}{4}\right)$

Superposition : $u = \sum_1^\infty a_n \sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi t}{4}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{n\pi t}{4}\right)$

3(b) $u(x,0) = f(x) = \sum_1^\infty a_n \sin\left(\frac{n\pi x}{2}\right) \Rightarrow a_n = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx \Big/ \int_0^2 \sin^2\left(\frac{n\pi x}{2}\right)dx$

$u_t(x,0) = g(x) = \sum_1^\infty b_n\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi x}{2}\right) \Rightarrow \frac{n\pi}{4} b_n = \int_0^2 g(x)\sin\left(\frac{n\pi x}{2}\right)dx \Big/ \int_0^2 \sin^2\left(\frac{n\pi x}{2}\right)dx$

answer: $a_n = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$, $b_n = \frac{4}{n\pi}\int_0^2 g(x)\sin\left(\frac{n\pi x}{2}\right)dx$

3(c) $a_n = \int_0^2 100\sin\left(\frac{n\pi x}{2}\right)dx = \dfrac{1 - \cos(n\pi)}{n\pi/2}$, $b_n = 0$

Use this page to start your solution. Attach any extra pages, then re-staple.

4. (CH H4. Rectangular Membrane)
Complete the following for the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) = c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a,\quad 0<y<b,\quad t>0,\\ u(x,y,t) = 0 & \text{on the boundary,}\\ u(x,y,0) = f(x,y), & 0<x<a,\quad 0<y<b,\\ u_t(x,y,0) = g(x,y), & 0<x<a,\quad 0<y<b. \end{cases}$$

under the assumptions $a=b=c=1$, $f(x,y)=1$, $g(x,y)=0$.

(1) Separation of variables.
(2) Product solution boundary value problems.
(3) Product solutions (the normal modes).
(4) Superposition for $u(x,y,t)$.
(5) Generalized Fourier coefficient formulas.
(6) Explicit numerical values for the coefficients.

(1) $u = X(x)Y(y)T(t) \Rightarrow \dfrac{T''}{c^2T} \overset{?}{=} \dfrac{X''}{X} + \dfrac{Y''}{Y} = -\lambda$, $\dfrac{X''}{X} = -\lambda - \dfrac{Y''}{Y} = -\mu$

(2) $\begin{cases} X''+\mu X = 0 \\ X(0)=X(1)=0 \end{cases}$ $\begin{cases} Y''+(\lambda-\mu)Y = 0 \\ Y(0)=Y(1)=0 \end{cases}$ $\begin{cases} T''+c^2\lambda T = 0 \\ T \neq 0 \end{cases}$

From the X-problem, $\mu>0$ and $1\cdot\sqrt{\mu}=n\pi$, $X(x)=\sin(n\pi x)$
From the Y-problem, $\lambda-\mu>0$; let $p=\lambda-\mu>0$, then $1\cdot\sqrt{p}=m\pi$ and
$Y=\sin(m\pi y)$. Finally, $\lambda=\mu+p=(n\pi)^2+(m\pi)^2$, then
$T(t) = a_{mn}\cos(\sqrt{\lambda_{m,n}}\,ct)+b_{mn}\sin(\sqrt{\lambda_{mn}}\,ct)$, $\sqrt{\lambda_{mn}}=\sqrt{(n\pi)^2+(m\pi)^2}$,
$n=1,..\infty$, $m=1,..\infty$.

(3) Normal modes: $\sin(n\pi x)\sin(m\pi y)\cos(\sqrt{\lambda_{mn}}\,ct)$,
$\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{\lambda_{mn}}\,ct)$.

(4) Add, $u(x,y,t) = \sum_1^\infty\sum_1^\infty a_{mn}\sin(n\pi x)\sin(m\pi y)\cos(\sqrt{\lambda_{mn}}\,ct) + b_{mn}\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{\lambda_{mn}}\,ct)$

(5) $f(x,y)=u(x,y,0)=\sum\sum a_{mn}\sin(n\pi x)\sin(m\pi y) \Rightarrow a_{mn}\int_0^1\int_0^1 \sin^2(n\pi x)\sin^2(m\pi y)\,dx\,dy$
$= \int_0^1\int_0^1 f(x,y)\sin(n\pi x)\sin(m\pi y)\,dx\,dy$ and
$g(x,y)=u_t(x,y,0)=\sum\sum b_{mn}(\sqrt{\lambda_{mn}}\,c)\sin(n\pi x)\sin(m\pi y) \Rightarrow$
$b_{mn}(c\sqrt{\lambda_{mn}})\int_0^1\int_0^1 \sin^2(n\pi x)\sin^2(m\pi y)\,dx\,dy = \int_0^1\int_0^1 g(x,y)\sin(n\pi x)\sin(m\pi y)\,dx\,dy$

(6) $b_{mn}=0$ when $g(x,y)=0$. when $f(x,y)=1$, then

$$a_{mn} = \frac{\int_0^1\int_0^1 \sin n\pi x\,\sin m\pi y\,dx\,dy}{\frac{1}{2}\cdot\frac{1}{2}} = 4\left(\frac{1-\cos(n\pi)}{n\pi}\right)\left(\frac{1-\cos(m\pi)}{m\pi}\right)$$

Use this page to start your solution. Attach any extra pages, then re-staple.

5. (CH H5. Heat Equation and Gauss' Heat Kernel)

5(a) [10%] Define a Fourier transform pair so as to include as many definitions as possible, in particular, Haberman's definition.

5(b) [20%] Assume $f(x) = 100$ on $-1 \le x \le 1$ and $f(x) = 0$ otherwise. Compute the Fourier transform $F(w)$ of $f(x)$.

Graded Details: (1) Transform formula, (2) Integration details, (3) Answer.

5(c) [50%] The heat kernel $g(x)$ and the error function $\mathbf{erf}(x)$ are defined by the equations

$$g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}, \quad \mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

Solve the infinite rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \frac{1}{16} u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Graded Details: (1) Fourier Transform method, (2) Convolution, (3) Heat kernel use, (4) Error function methods, (5) Final answer, expressed in terms of the error function.

5(d) [10%] Compute the limits at x equal to infinity and minus infinity of $u(x,t)$. zero

5(e) [10%] Compute the limit $u(x, 0+)$ for each x in $-\infty < x < \infty$. 0, 50, 100, 75, 50, 25, 0

5(a) $FT[f(x)] = cv \int_{-\infty}^{\infty} f(x)\, e^{i w x s}\, dx$, $cv = \frac{1}{2\pi}$, $s=1$ for Haberman

$f(x) = \int_{-\infty}^{\infty} F(w)\, e^{-iwxs}\, dw$, $s=1$ for Haberman, $F(w) = FT[f(x)]$

5(b) $F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{iwx} dx = \frac{1}{2\pi}\int_{-1}^{1} 100 e^{iwx} dx = \frac{1}{2\pi}(100)\left(\frac{e^{iw} - e^{-iw}}{iw}\right)$

5(c) $FT[u_t] = \frac{1}{16} FT[u_{xx}] \Rightarrow \frac{d}{dt} U = \frac{1}{16}(-iw)^2 U \Rightarrow U = U_0 e^{-\frac{w^2 t}{16}}$ and $U_0 = F(w)$. Assume $G(w) = FT[g(x)] = e^{-w^2 t/16}$. Then $FT[u(x,t)] = U = F(w)\, G(w) \Rightarrow$ by convolution that $u(x,t) = f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv$. Comparing the heat kernel identity, then $k = \frac{1}{16}$ and $g(x) = 4\sqrt{\frac{\pi}{t}}\, e^{-4x^2/t}$

Then $u(x,t) = \frac{4\sqrt{\pi}}{2\pi\sqrt{t}} \int_{-\infty}^{\infty} f(v)\, e^{-4(x-v)^2/t} dv = \frac{2}{\sqrt{\pi t}} \int_0^1 50 e^{-\frac{4(x-v)^2}{t}} dv$

$+ \frac{2}{\sqrt{\pi t}} \int_{-1}^{0} 100\, e^{-4\frac{(x-v)^2}{t}} dv$. Calculus details on the next page, $z = \frac{2(v-x)}{\sqrt{t}}$:

$u(x,t) = \left(\frac{1}{\sqrt{t}}(50)\frac{2}{\sqrt{\pi}}\int_{v_1}^{v_2} e^{-z^2} dz + \frac{1}{\sqrt{t}}(100)\frac{2}{\sqrt{\pi}}\int_{v_3}^{v_1} e^{-z^2} dz\right) \cdot \frac{\sqrt{t}}{2}$

$v_1 = \frac{2(0-x)}{\sqrt{t}}$, $v_2 = \frac{2(1-x)}{\sqrt{t}}$, $v_3 = \frac{2(-1-x)}{\sqrt{t}}$

$u(x,t) = \frac{1}{2}(\mathrm{erf}(v_2) - \mathrm{erf}(v_1))50 + \frac{1}{2}(\mathrm{erf}(v_1) - \mathrm{erf}(v_3))100$ (next page)

Calculus details 5(c)

$$z = \frac{2(v-x)}{\sqrt{t}}, \quad dz = \frac{2}{\sqrt{t}}\,dv$$

$$\int_0^1 e^{-\frac{4(x-v)^2}{t}}\,dv = \int_{v_1}^{v_2} e^{-z^2}\frac{\sqrt{t}}{2}\,dz$$

where $v_1 = \left.\frac{2(v-x)}{\sqrt{t}}\right|_{x=0} = \frac{2(0-x)}{\sqrt{t}}$

$v_2 = \left.\frac{2(v-x)}{\sqrt{t}}\right|_{x=1} = \frac{2(1-x)}{\sqrt{t}}$

Similarly,

$$\int_{-1}^{0} e^{-\frac{4(x-v)^2}{t}}\,dv = \int_{v_3}^{v_1} e^{-z^2}\frac{\sqrt{t}}{2}\,dz, \quad v_3 = \frac{2(-1-x)}{\sqrt{t}}$$

Then

$$u(x,t) = \frac{2}{\sqrt{\pi t}}\frac{\sqrt{t}}{2}\left(50\int_{v_1}^{v_2} e^{-z^2}dz + 100\int_{v_3}^{v_1} e^{-z^2}dz\right)$$

$$= \frac{50}{2}\frac{2}{\sqrt{\pi}}\int_{v_1}^{v_2} e^{-z^2}dz + \frac{100}{2}\frac{2}{\sqrt{\pi}}\int_{v_3}^{v_1} e^{-z^2}dz$$

$$= 25\big(\text{erf}(v_2) - \text{erf}(v_1)\big) + 50\big(\text{erf}(v_1) - \text{erf}(v_3)\big)$$

5(d) Because $\text{erf}(\infty) = 1$, $\text{erf}(-\infty) = -1$, then
$u(+\infty, t) = 25(\text{erf}(-\infty) - \text{erf}(-\infty)) + 50(\text{erf}(-\infty) - \text{erf}(-\infty)) = 0$
$u(-\infty, t) = 0$ similarly
For $x=10$, $u(10, 0+) = \lim_{t\to 0+} u(10,t) = 25(\text{erf}(-\infty) - \text{erf}(-\infty))$
$+50(\text{erf}(-\infty) - \text{erf}(-\infty)) = 0$. Similarly, $u(x, 0+) = 0$ for $1 < x < \infty$,
$-\infty < x < -1$. But $u(1, 0+) = 25(\text{erf}(-\infty) - \text{erf}(0)) + 50(\text{erf}(0) - \text{erf}(-\infty))$
$= 25$, $u(-1, 0+) = 50$, $u(0, 0+) = 75$, $u(x, 0+) = 50$ on $0 < x < 1$,
$u(x, 0+) = 100$ on $-1 < x < 0$. Exams use "find $u(1, 0+)$."