Math 3150-3 Review Sheet - Spring, 2007

In this rather short course, we introduced some partial differential equations (PDEs) modeled on certain natural phenomena (heat transfer, membrane vibration, etc.), and developed several approaches that are based on Fourier series and Fourier transform.

1 Partial Differential Equations

There are mainly three types of equations discussed this semester: wave equation, heat equation, and Laplace's equation. You may wonder why the first-order convection equation ($u_t + au_x = 0$) did not make to that list. The reason is that it is covered in some sense by the second-order wave equation which has been traditionally more famous.

One of the major differences between an ODE and a PDE, besides the extra variables, is the presence of the boundary conditions. A change in boundary shape and/or the boundary condition can lead to a completely different approach in solving the problem, often makes an analytic solution impossible. For this reason, we usually talk about a PDE problem, which specifies the initial and boundary conditions, as opposed to an equation.

The main approach we have developed in this course is the classic **separation of variables** technique. This technique follows the simple strategy that attempts to reduce a PDE problem to several ODE problems, for which we have learned many tools in Math 2250. The procedure to separate variables is quite straightforward and easy to follow. However, the success relies on two essential features of the problem: that the equation and boundary conditions are linear, and that the boundary conditions lead naturally to certain eigenfunctions. The latter part is especially difficult in many applications in that it is not clear that the eigenvalue problem can be easily solved.

The linearity of the equation and the boundary conditions in a PDE problem allows us to break a problem into several and solve them separately, if we happen to be able to do so. Typically we handle one non-homogeneous part at a time. For example, if we need to solve a problem with a non-zero term on the right-hand-side of the equation, and a non-zero boundary condition, we solve the problem with non-zero right-hand-side in the equation, with zero boundary conditions first, and then solve another problem with zero right-hand-side, but non-zero boundary conditions next. Then according to the **principle of superposition** we can add these two solutions to obtain the solution that solves our problem.

We have worked out several problems in this course, and we list those main solution features in the following tables. Note that they are all problems with certain zero boundary conditions. For non-zero boundary conditions, you will need to solve another problem (such as the steady state solution for the heat equation). On the other hand, the eigenfunction expansion approach can help us solve the problem with non-zero right-hand-sides. The factors listed can be viewed as building blocks to build solutions that solve your particular problems.

Table 1: One-dimensional time dependent problems

equation	boundary conditions	factor in x	factor in t
wave equation	$u(0,t) = u(L,t) = 0$	$\sin(\frac{n\pi x}{L})$	$\cos(\frac{cn\pi t}{L}), \sin(\frac{cn\pi t}{L})$
heat equation	$u(0,t) = u(L,t) = 0$	$\sin(\frac{n\pi x}{L})$	$e^{-(\frac{cn\pi}{L})^2 t}$

Table 2: Two-dimensional time dependent problems ($\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$)

equation	boundary conditions	factor in space variables	factor in t
wave equation	$u = 0$ on rectangular boundaries	$\sin(\frac{m\pi x}{a})\sin(\frac{m\pi y}{b})$	$\cos\lambda_{mn}t, \sin\lambda_{mn}t$
heat equation	$u = 0$ on rectangular boundaries	$\sin(\frac{m\pi x}{a})\sin(\frac{m\pi y}{b})$	$e^{-\lambda_{mn}^2 t}$
wave equation	$u = 0$ on $r = a$	$\cos(c\lambda_n t), \sin(c\lambda_n t)$	$J_0(\lambda_n r),\ \lambda_n = \alpha_n/a$

Table 3: Two-dimensional Laplace's equation

shape of region	boundary conditions	factor in x	factor in y
rectangular	$u(0,y) = u(a,y) = u(x,b) = 0$	$\sin(\frac{n\pi x}{a})$	$\sinh(\frac{n\pi(b-y)}{a})$
rectangular	$u(0,y) = u(a,y) = u(x,0) = 0$	$\sin(\frac{n\pi x}{a})$	$\sinh(\frac{n\pi y}{a})$
rectangular	$u(a,y) = u(x,0) = u(x,b) = 0$	$\sinh(\frac{n\pi(a-x)}{b})$	$\sin(\frac{n\pi y}{b})$
rectangular	$u(0,y) = u(x,0) = u(x,b) = 0$	$\sinh(\frac{n\pi x}{b})$	$\sin(\frac{n\pi y}{b})$
circular	$u(0,\theta)$ is finite	$r^n,\ n > 0$	$\sin(n\theta),\ \cos(n\theta)$
wedge	$u(r,0) = u(r,\alpha) = 0$	$r^{\frac{n\pi}{\alpha}}$	$\sin(\frac{n\pi\theta}{\alpha})$

You should be careful with the boundary conditions specified in the problem. A change in the boundary conditions means that you will have to modify the basic solution factors listed above.

Several features deserve special mention: for time dependent problems (wave equation and heat equation), the behavior in t is quite different. In wave equations, if you fix at a point and observe the change in t, you will find the sin or cos waves which move in a periodic fashion and keep returning. On the other hand, the heat equation solutions show a quite different behavior in t: the exponential decay in time. Eventually everything will decay to zero if zero boundary conditions are specified, or to whatever steady solution if non-zero boundary conditions are given.

Finally we will use these building blocks to obtain the solution that solves the problem and the last step is usually to choose the right combination to match the initial conditions, or the non-zero part of the boundary conditions. For this step, we need the tools of Fourier series.

2 Fourier Series

The idea of using a series to represent a function is to establish an identification of the function through a sequence of numbers. This is similar to the idea of representing a vector in space by an ordered set of coordinates. Each term in the series is like a vector along a certain direction so the combination can point wherever you want, that is whatever function you want to address. There are many technical questions such as whether you can really represent all the functions you want to consider, and if the series converges. This is why we need to verify those conditions stated in the theorem before we can start. In the case of Fourier series, we try to represent all these piecewise smooth functions (meaning finite amplitude jumps are allowed in the function under consideration) defined over a finite interval. The Fourier series obtained, through determination of the Fourier coefficients, is a periodic function which means that it is defined over $(-\infty, \infty)$, while the original function is probably just defined over a finite interval. This is fine since we only care about that interval, and as long as these two functions agree in that interval we can use the series to replace the original function. So different functions can be represented by different collections of coefficients.

The rest of the work is the determination of the coefficients. We are given all the formulas and the work is done by computing various integrals involving sin and cos. Quite often we need to use the technique of **integration by parts**. One important property of trigonometry functions is the orthogonality condition:

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0, \quad \text{if } m \neq n,$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0, \quad \text{for all } m \text{ and } n,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0, \quad \text{if } m \neq n.$$

This allows us to have these nice formulas to determine the coefficients, it also reduces the determination problem to a trivial one if the function to be represented is already a trigonometry function with the right frequency. To see this, consider the problem of finding coordinates for vectors in a plane. A vector with length one along the 45° direction is represented by $(1/\sqrt{2}, 1/\sqrt{2})$ after a short computation. On the other hand, a vector with length 3 along the positive x-axis is readily represented by $(3, 0)$, which does not involve any computation. The situation is similar in Fourier coefficients: if the function is already one of these sin or cos multiplied by a constant, that constant is the coefficient corresponding to that term, and you can simply write it down without actually going through the integral.

The connection between the PDE problem and Fourier series is that we will represent our solution in a Fourier series, therefore the determination of the solution is equivalent to the determination of the Fourier coefficients.

3 Fourier Transform

One major limitation of the Fourier series is that they are designed to represent functions over a finite interval, or periodic functions. If we want to work with non-periodic functions defined over $(-\infty, \infty)$, which turn up in PDE problems over infinite spatial intervals, we need to use the Fourier transform. Again, there are technical conditions whether the Fourier transform for your function exists or not. The main condition is that the function itself has to be integrable:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

The obvious advantage of Fourier transform is that it handles derivatives really well: it translates differentiation in the x-space to multiplication by $i\omega$ in the Fourier space. The differential equation is therefore greatly simplified after the Fourier transform. However, we need to pay a price that the final solution will have to be represented by an integral. Sometimes we get lucky that the integral can be explicitly evaluated. In many cases we just have to live with the reality that the solution is represented by an integral with x and t as parameters. There are many techniques and tricks available to obtain the inverse transform, that is to work out the integral. But we choose, due to limited time available in the course, not to devote much energy in mastering the skill. Nevertheless, we need to understand the basic concept and focus on the identification of variables in various integral expressions.

Finally we want to emphasize the importance of the heat kernel

$$K(x) = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2t}},$$

and the convolution $K * f$ as the solution to the initial value problem for the heat equation over $(-\infty, \infty)$:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) K(x-y) dy.$$