Math 3150-3 Final, May 1, 2007

Name: ____________________ **U. ID:** ____________

Instructions: This is a closed book but open written notes exam. Calculators are not allowed. You need to show all the details of your work to receive full credits.

Problem	1	2	3	4	5	6	7	8	total
worth of points	10	18	10	12	15	15	10	10	100
your points									

1. The smallest integer function, also called the ceiling function, is defined as

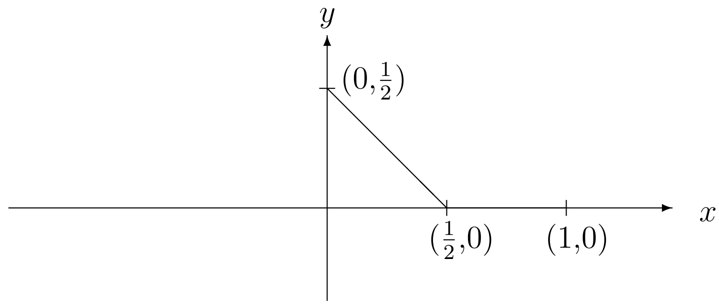
$$\lceil x \rceil = \text{smallest integer that is larger than or equal to } x$$

For example, $\lceil 3.1 \rceil = 4$, $\lceil 3 \rceil = 3$, and $\lceil -3.1 \rceil = -3$.

(a) Plot this function for $x \in [-3, 3]$. Make sure that the discontinuities are accurately described. Is this function a periodic function?

(b) Plot the function $f(x) = x - \lceil x \rceil$. Is $f(x)$ periodic? What is the period of this function if it is periodic?

2. The following function $g(x)$ is defined for $x \in (0, 1)$:



(a) Suppose $f(x)$ is a periodic function with period 1 which agrees with $g(x)$ when $x \in (0, 1)$. Plot $f(x)$ for $x \in (-\frac{1}{2}, \frac{1}{2})$ and find its Fourier series representation.

(b) Now let $h(x)$ be the sine series expansion of $g(x)$, plot $h(x)$ for $x \in (-1, 1)$. What is the value of p in this case? Find this sine series expansion of g.

3. Solve the following wave equation problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0.$$

$$u(0,t) = 0, \ u(1,t) = 0,$$

$$u(x,0) = 0, \ \frac{\partial u}{\partial t}(x,0) = g(x).$$

Where $g(x)$ is the function plotted in Problem 2. Hint: you should use the result of Problem 2 to avoid repeating calculations.

4. Solve the following heat equation problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \; t > 0,$$

with boundary conditions

$$u(0,t) = 10, \quad u(1,t) = 0,$$

$$u(x,0) = 10(1-x) + 2\sin 2\pi x, \quad 0 < x < 1.$$

5. Solve the heat equation in a unit square:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < 1,\ 0 < y < 1,\ \text{and } t > 0,$$

with boundary conditions

$$u(0, y, t) = u(1, y, t) = 0 \text{ for } 0 \le y \le 1 \text{ and } t \ge 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0 \text{ for } 0 \le x \le 1 \text{ and } t \ge 0,$$

and initial condition

$$u(x, y, 0) = \begin{cases} 100 & \text{if } 0 < x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

6. Solve the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin(\pi x)\sin(\pi y), \quad 0 < x < 1,\ 0 < y < 1,$$

with boundary conditions $u(x,0) = f_1(x) = 4\sin(\pi x),\ f_2(x) = g_1(y) = g_2(y) = 0.$

7. Solve the Dirichlet problem for the Laplace's equation on the unit disk, in polar coordinates, for the given boundary value

$$u(1,\theta) = f(\theta) = \begin{cases} 0 & \text{if } 0 \le \theta < \pi/2, \\ 1 & \text{if } \pi/2 \le \theta \le \pi, \\ 0 & \text{if } \pi < \theta < 2\pi. \end{cases}$$

8. Use Fourier transform to solve the heat equation problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \ t > 0,$$

with the initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } -2 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What can you say about the boundary value $\lim_{x \to \pm\infty} u(x,t)$?