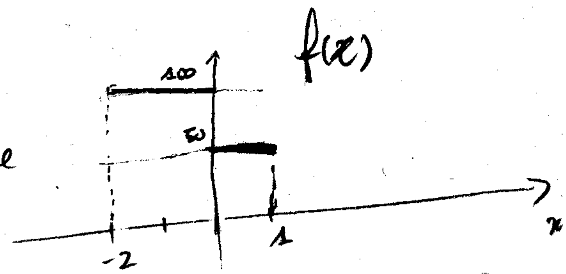
HW9 Math 3150-1

7.4.2

$$\begin{cases} u_t = \frac{1}{100} u_{xx} \\ u(x,0) = f(x) \end{cases}$$

where



Using theorem 7.4.1:

$$\boxed{u(x,t) =} \frac{100}{\sqrt{2t}} e^{-x^2 100^2/4t} * f = \frac{100}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s)\, e^{-(x-s)^2 100^2/4t}\, ds$$

$$= \frac{100^2}{2\sqrt{\pi t}} \int_{-2}^{0} e^{-(x-s)^2 100^2/4t}\, ds + \frac{100 \cdot 50}{2\sqrt{\pi t}} \int_{0}^{1} e^{-(x-s)^2 100^2/4t}\, ds$$

$$= -\frac{100^2}{2\sqrt{\pi t}} \frac{\sqrt{t}}{50} \int_{50(x+2)/\sqrt{t}}^{50x/\sqrt{t}} e^{-u^2}\, du - \frac{100 \cdot 50}{2\sqrt{\pi t}} \frac{\sqrt{t}}{50} \int_{50x/\sqrt{t}}^{50(x-1)/\sqrt{t}} e^{-u^2}\, du$$

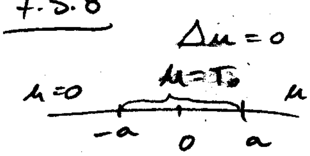
$$u = \frac{(x-s)100}{2\sqrt{t}}$$

$$du = -\frac{50\, ds}{\sqrt{t}}$$

$$= -\frac{100}{\sqrt{\pi}}\left(\operatorname{erf}\left(\frac{50(x+2)}{\sqrt{t}}\right) - \operatorname{erf}\left(\frac{50x}{\sqrt{t}}\right)\right) - \frac{50}{\sqrt{\pi}}\left(\operatorname{erf}\left(\frac{50(x-1)}{\sqrt{t}}\right) - \operatorname{erf}\left(\frac{50x}{\sqrt{t}}\right)\right)$$

$$\boxed{= \frac{50}{\sqrt{\pi}}\left[-2\operatorname{erf}\left(\frac{50(x+2)}{\sqrt{t}}\right) + \operatorname{erf}\left(\frac{50x}{\sqrt{t}}\right) + \operatorname{erf}\left(\frac{50(x-1)}{\sqrt{t}}\right)\right]}$$

7.5.8



Theorem 7.5.1

$$\boxed{u(x,y) =} \frac{T_0\, y}{\pi} \int_{-a}^{a} \frac{1}{(x-s)^2 + y^2}\, ds$$

$$= \frac{T_0}{\pi y} \int_{-a}^{a} \frac{ds}{1 + (x-s)^2/y^2}$$

$$= \frac{T_0}{\pi}\left[\operatorname{atan}\frac{a-x}{y} - \operatorname{atan}\frac{-a-x}{y}\right]$$

$$\boxed{= \frac{T_0}{\pi}\left[\operatorname{atan}\frac{a-x}{y} + \operatorname{atan}\frac{a+x}{y}\right]}$$

This is a generalization of Example 7.5.1 because it allows for a different temperature at the boundary and a different interval where temp is $\neq 0$.

7.5.9 The isotherms $u(x,y) = T$, $0 < T < T_0$ are points (x,y) s.t. ②

$$\operatorname{atan}\left(\frac{a+x}{y}\right) + \operatorname{atan}\left(\frac{a-x}{y}\right) = \frac{\pi T}{T_0}$$

Taking tangents on both sides and using $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$:

$$\frac{\frac{a+x}{y} + \frac{a-x}{y}}{1 - \frac{(a+x)}{y}\frac{(a-x)}{y}} = \frac{2ay}{x^2+y^2-a^2} = \tan\left(\frac{\pi T}{T_0}\right)$$

$$\Rightarrow \quad x^2+y^2 - \frac{2ay}{\tan(\pi T/T_0)} = a^2$$

$$\boxed{x^2 + \left(y - \frac{a}{\tan(\pi T/T_0)}\right)^2 = a^2 + \frac{a^2}{(\tan(\pi T/T_0))^2} = \left(\frac{a}{\sin(\pi T/T_0)}\right)^2}$$

which are the circles of center $\left(0, \dfrac{a}{\tan(\pi T/T_0)}\right)$ and radius $\dfrac{a}{\sin(\pi T/T_0)}$.

when $T = \dfrac{T_0}{2}$

$$x^2 + \left(y - \frac{a}{\tan(\pi/2)}\right)^2 = \left(\frac{a}{\sin(\pi/2)}\right)^2 \iff \boxed{x^2+y^2 = a^2}$$

7.6.4 $f(x) = x^2 e^{-x^2}$

$$\boxed{\mathcal{F}_c(x^2e^{-x^2})} = \frac{d}{d\omega}\mathcal{F}_s(xe^{-x^2}) = -\frac{d^2}{d\omega^2}\underbrace{\mathcal{F}_c(e^{-x^2})}_{= e^{-\omega^2}} = -\frac{d^2}{d\omega^2}e^{-\omega^2} = \boxed{(2-4\omega^2)e^{-\omega^2}}$$

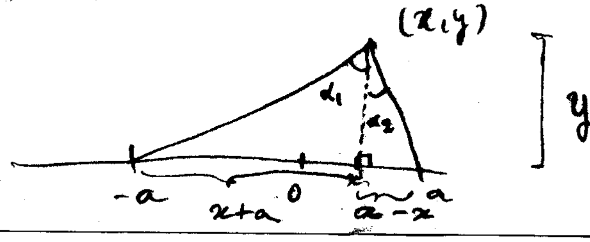
because e^{-x^2} is an even fun.

7.6.8

$$\boxed{\mathcal{F}_s(xe^{-x^2}) = -\frac{d}{d\omega}\mathcal{F}_c(e^{-x^2}) = -\frac{d}{d\omega}e^{-\omega^2} = 2\omega e^{-\omega^2}}$$

7.5.10 (EXTRA CREDIT)

(a) $\boxed{\alpha(x,y) = \alpha_1(x,y) + \alpha_2(x,y)} = \operatorname{atan}\left(\frac{y}{x+a}\right) + \operatorname{atan}\left(\frac{y}{a-x}\right) = \boxed{\frac{\pi}{T_0}u(x,y)}$



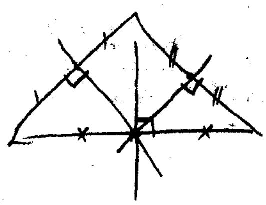
7.5.10 (cont'd)

(a) Thus the temperature at every point (x,y) is proportional to the angle

$\widehat{(-a,0),\,(x,y),\,(a,0)}$.

(b) The isotherms are the sets of points (x,y) for which this angle is constant. So they must be circles passing through $(-a,0)$ and $(a,0)$. Given T this angle is $\pi \dfrac{T}{T_0}$.

To construct the corresponding isotherm simply construct the triangle $(-a,0)$ $(a,0)$, $\left(0, a\times\tan\frac{\pi T}{2T_0}\right)$ and its circumcircle is the isotherm (the center of the circumcircle is the intersection of the perpendicular bisectors of a triangle.



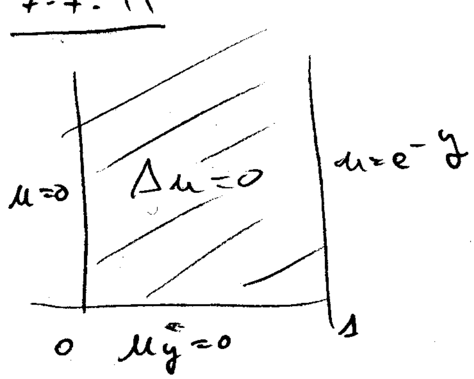
7.7.2

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = 0 \end{cases} \quad , x>0, t>0 \quad , \text{and } f(x) = \begin{cases} \sin x & \text{if } 0<x<\pi \\ 0 & \text{otherwise} \end{cases}$$

Following example 7.7.1

$$\boxed{\hat{u}_s(\omega,t) = \hat{f}_s(\omega)\, e^{-\omega^2 t} = \underbrace{\sqrt{\frac{2}{\pi}}\,\frac{\sin(\pi\omega)}{1-\omega^2}}_{\text{using example 7.6.1}}\, e^{-\omega^2 t}}$$

7.7.11



We solve problem

$$\begin{cases} \Delta u = 0 & \text{of } x\in(0,1),\ y>0 \\ u(0,y) = 0, \\ u_y(x,0) = 0 \\ u(1,y) = e^{-y} \end{cases}$$

Taking $\mathcal{F}_c$ of the DE w.r.t. y:

$$\frac{d^2}{dx^2}\hat{u}_c(x,\omega) - \omega^2 \hat{u}_c(x,\omega) - \sqrt{\frac{2}{\pi}}\underbrace{u_y(x,0)}_{=0} = 0$$

General solution is:

$$\hat{u}_c(x,\omega) = A(\omega)\cosh \omega x + B(\omega)\sinh \omega x$$

Since:

$$\hat{u}_c(0,\omega) = 0 \Rightarrow A(\omega) = 0$$

$$\hat{u}_c(1,\omega) = B(\omega)\sinh \omega = \mathcal{F}_c(e^{-y}) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$$

$$\Rightarrow \boxed{\hat{u}_c(x,\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}\frac{\sinh \omega x}{\sinh \omega}}$$

$$\Rightarrow \quad u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\frac{\sinh \omega x}{\sinh \omega}\cos \omega y \, d\omega$$