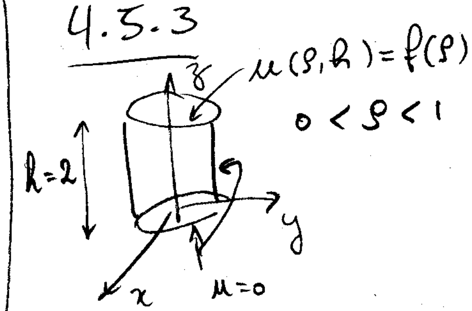
HW7 Solutions

4.5.3



Solve $\Delta u=0$ with $f(\rho)=\begin{cases}100 & \text{if } 0<\rho<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}<\rho<1\end{cases}$

The PDE is:

$$\begin{cases}\Delta u=0\\ u(\rho,0)=0\\ u(1,z)=0\\ u(\rho,2)=f(\rho)\end{cases}$$

Solution is of the form:

$$u(\rho,z)=\sum_{n=1}^{\infty} A_n J_0(\lambda_n\rho)\sinh\lambda_n z$$

where $\lambda_n=\frac{\alpha_n}{1}=$ n-th zero of 0-th order Bessel function $J_0(\cdot)$.

and

$$A_n=\frac{2}{\sinh(2\alpha_n)J_1^2(\alpha_n)}\int_0^1 f(\rho)J_0(\alpha_n\rho)\rho\,d\rho$$

$$=\left(\quad '' \quad\right)100\int_0^{\frac{1}{2}} J_0(\alpha_n\rho)\rho\,d\rho \qquad \begin{aligned}x&=\alpha_n\rho\\ dx&=\alpha_n d\rho\end{aligned}$$

$$=\frac{200}{\sinh(2\alpha_n)J_1^2(\alpha_n)}\int_0^{\frac{\alpha_n}{2}} J_0(x)\frac{x\,dx}{\alpha_n^2}$$

$$=\frac{200}{\sinh(2\alpha_n)J_1^2(\alpha_n)\alpha_n^2}\; xJ_1(x)\Big|_0^{\frac{\alpha_n}{2}}$$

$$=\boxed{\frac{100\,J_1\left(\frac{\alpha_n}{2}\right)}{\sinh(2\alpha_n)J_1^2(\alpha_n)\alpha_n}}$$

7.1.2 Find the Fourier integral representation of the function:

$$f(x) = \begin{cases} -1 & \text{if } -1<x<0 \\ 1 & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

We have $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt.$

Here $f(-x) = -f(x) \Rightarrow f$ is an odd function $\Rightarrow \boxed{A(\omega) = 0}$

and: $\boxed{B(\omega)} = \frac{1}{\pi}\int_{-1}^{1} f(t)\sin\omega t\, dt = \frac{2}{\pi}\int_0^1 \sin\omega t\, dt$

$$= \frac{-2}{\pi\omega}\cos\omega t\Big|_0^1 = \boxed{\frac{2(1-\cos\omega t)}{\pi\omega}}$$

$$\Rightarrow \boxed{f(x) = \int_0^{\infty} d\omega \cos\omega x\, \frac{2(1-\cos\omega t)}{\pi\omega}}$$

7.2.4

$$f(x) = \begin{cases} x & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\hat{f}(\omega)} = \int_{-\infty}^{\infty} dx\, e^{-i\omega x} f(x) = \int_{-1}^{1} x e^{-i\omega x}\, dx$$

$$\overset{IBP}{=} x\frac{e^{-i\omega x}}{-i\omega}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-i\omega x}}{-i\omega}\, dx$$

$$= \frac{-e^{-i\omega} + e^{+i\omega}}{i\omega} - \frac{e^{-i\omega x}}{(-i\omega)^2}\Big|_{-1}^{1} = -\frac{\cos\omega}{i\omega} + \frac{e^{-i\omega} - e^{i\omega}}{\omega^2}$$

$$\boxed{= i\left(\frac{\cos\omega}{\omega} - 2\frac{\sin\omega}{\omega^2}\right)}$$

7.2.20 Shifting in ω:

(3)

(a) $$\boxed{\mathcal{F}(e^{iax}f(x))(\omega)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i\omega x} e^{iax} f(x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ix(\omega-a)} f(x)$$

$$\boxed{= \hat{f}(\omega-a)}$$

Similarly: $\mathcal{F}(e^{-iax}f(x))(\omega) = \hat{f}(\omega+a)$

(b) $$\mathcal{F}(\cos(ax)f(x))(\omega) = \mathcal{F}\left(\frac{e^{iax}+e^{-iax}}{2} f(x)\right)(\omega)$$

$$= \frac{1}{2}\mathcal{F}(e^{iax}f(x))(\omega) + \frac{1}{2}\mathcal{F}(e^{-iax}f(x))(\omega)$$

$$= \frac{\hat{f}(\omega-a) + \hat{f}(\omega+a)}{2}$$

In the same way using Euler's formula:

$$\boxed{\mathcal{F}(\sin(ax)f(x))(\omega)} = \mathcal{F}\left(\frac{e^{iax}-e^{-iax}}{2i} f(x)\right)$$

$$= \boxed{\frac{\hat{f}(\omega-a) - \hat{f}(\omega+a)}{2i}}$$

For the two last relations one can use the result from 7.2.19:

$$\boxed{\mathcal{F}^{-1}(\cos(a\omega)\hat{f}(\omega))} = \mathcal{F}^{-1}\left(\frac{e^{ia\omega}+e^{-ia\omega}}{2}\hat{f}(\omega)\right)$$

$$\boxed{= \frac{f(x-a)+f(x+a)}{2}}$$

$$\boxed{\mathcal{F}^{-1}(\sin(a\omega)\hat{f}(\omega))} = \mathcal{F}^{-1}\left(\frac{e^{ia\omega}-e^{-ia\omega}}{2i}\hat{f}(\omega)\right)$$

$$= \boxed{\frac{f(x+a)-f(x-a)}{2i}}$$

7.2.23 $f(x) = \dfrac{\cos x + \cos 2x}{1+x^2}$

$$\hat{f}(\omega) = \widehat{\frac{\cos x}{1+x^2}}(\omega) + \widehat{\frac{\cos 2x}{1+x^2}}(\omega)$$

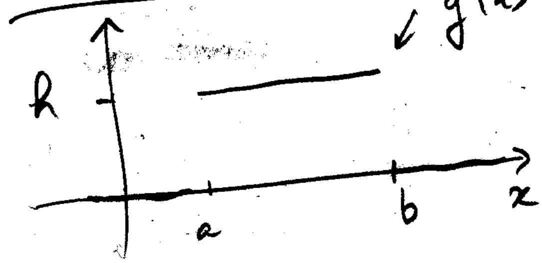
and $\widehat{\dfrac{1}{1+x^2}}(\omega) = \sqrt{\dfrac{\pi}{2}}\, e^{-|\omega|}$

~~thus~~:

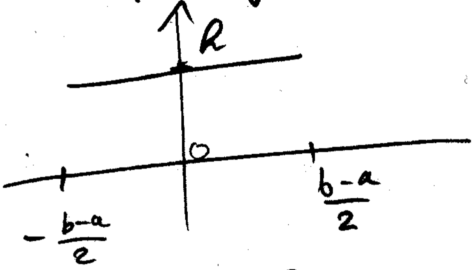
$$\boxed{\hat{f}(\omega) = \sqrt{\frac{\pi}{2}}\left[\frac{1}{2}\left(e^{-|\omega-1|} + e^{-|\omega+1|}\right) + \frac{1}{2}\left(e^{-|\omega-2|} + e^{-|\omega+2|}\right)\right]}$$

(using shifting result of problem 7.2.20

7.2.27 $g(x) = \begin{cases} h & \text{if } a<x<b \\ 0 & \text{otherwise} \end{cases}$



$f(x) = g\left(x - \dfrac{b+a}{2}\right) = h\begin{cases} 1 & \text{if } |x| < \dfrac{b-a}{2} \\ & \end{cases}$

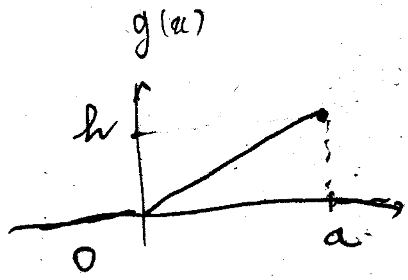


From example 7.2.1: $\hat{f}(\omega) = h\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin\left(\frac{b-a}{2}\omega\right)}{\omega}$

Thus from 7.2.20:

$$\boxed{\hat{g}(\omega) = h\sqrt{\frac{2}{\pi}}\, e^{-i\left(\frac{b+a}{2}\right)\omega}\,\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\omega}}.$$

7.2.29



$$g(x) = \frac{x}{a} f(x) \quad \text{where} \quad f(x) = \begin{cases} h & \text{if } 0 \le x \le a \\ 0 & \text{otherwise.} \end{cases}$$

from 7.2.27 we have:

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\, h\, e^{-i\frac{a}{2}\omega} \frac{\sin\left(\frac{a\omega}{2}\right)}{\omega}$$

thus:

$$\hat{g}(\omega) = \mathcal{F}\left(\frac{x}{a} f(x)\right)(\omega)$$

$$= \frac{i}{a}\frac{d}{d\omega}\hat{f}(\omega)$$

$$= i\frac{h}{a}\sqrt{\frac{2}{\pi}}\frac{d}{d\omega}\left[e^{-i\frac{a}{2}\omega}\frac{\sin\frac{a\omega}{2}}{\omega}\right]$$

$$= \frac{ih}{a}\sqrt{\frac{2}{\pi}}\left[-\frac{ia}{2}e^{-i\frac{a}{2}\omega}\frac{\sin\frac{a}{2}\omega}{\omega} + e^{-i\frac{a}{2}\omega}\frac{\frac{a}{2}\omega\cos\frac{a}{2}\omega - \sin\frac{a}{2}\omega}{\omega^2}\right]$$

$$= i\frac{h}{a}\sqrt{\frac{2}{\pi}}\frac{e^{-i\frac{a}{2}\omega}}{\omega}\left[\frac{a}{2}\left[\cos\frac{a\omega}{2} - i\sin\frac{a\omega}{2}\right] - \frac{\sin\left(\frac{a\omega}{2}\right)}{\omega}\right]$$

$$\boxed{= i\frac{h}{a}\sqrt{\frac{2}{\pi}}\frac{e^{-i\frac{a}{2}\omega}}{\omega}\left[\frac{a}{2}e^{-\frac{ia\omega}{2}} - \frac{\sin\frac{a\omega}{2}}{\omega}\right]}$$