4.1.2: $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \theta$

thus $\boxed{\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0}$

4.1.3 $u(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

$\Rightarrow u_r = -\frac{1}{r^2}$ thus $\boxed{\Delta u = \frac{2}{r^3} - \frac{1}{r^3} = \frac{1}{r^3} \neq 0}$

$u_{rr} = +\frac{2}{r^3}$

4.2.2 We want to solve the problem:

$$\begin{cases} u_{tt} = 10\left(u_{rr} + \frac{1}{r} u_r\right) \\ u(r,0) = 1 - r^2 \\ u_t(r,0) = 1 \\ u(1,t) = 0 \end{cases}$$

We first need to compute A_n and B_n as in Theorem 1 of this chapter.

For A_n we use method of example 2:

$$\boxed{A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1-r^2) J_0(\alpha_n r)\, r\, dr} \overset{s = \alpha_n r}{=} \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s)\, s\, ds$$

$$\overset{IBP}{=} \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \left[\underbrace{\cancel{(\alpha_n^2 - s^2) J_1(s) s \Big|_0^{\alpha_n}}}_{0} + 2 \int_0^{\alpha_n} J_1(s) s^2\, ds \right]$$

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2\, ds = \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \alpha_n^2 J_2(s) \Big|_0^{\alpha_n} = \boxed{\frac{4 J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)}}$$

or with more simplification: $\boxed{A_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)}}$ See example 4.2.2

②

<u>Note</u> : the identities we used here are

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \Rightarrow \int x^p J_{p-1}(x)\,dx = x^p J_p(x)$$

$$(1) \quad \int x J_0(x)\,dx = x J_1(x).$$

and:

$$(2) \quad \int x^2 J_1(x)\,dx = x^2 J_2(x)$$

Now for the B_n we have:

$$\boxed{\frac{\alpha_n c}{1} B_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r)\, r\,dr} = \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_0(s)\, s\,ds$$

$$= \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} \, s J_1(s) \Big|_0^{\alpha_n} = \boxed{\frac{2}{\alpha_n J_1(\alpha_n)}}$$

with $c = 10$:

$$\boxed{B_n = \frac{1}{5\alpha_n^2 J_1(\alpha_n)}}$$

Thus $\boxed{u(r,t) = \sum_{n=1}^{\infty} J_0(\alpha_n r) \left[\frac{8}{\alpha_n^3 J_1(\alpha_n)} \cos(10\alpha_n t) + \frac{1}{5\alpha_n^2 J_1(\alpha_n)} \sin(10\alpha_n t) \right]}$

<u>4.2.4</u> We would like to solve the problem:

$$\begin{cases} u_{tt} = u_{rr} + \frac{1}{r} u_r \\ u(r,0) = 0 \\ u_t(r,0) = J_0(\alpha_3 r) \\ u(1,t) = 0 \end{cases}$$

Since $u(r,0)$: $\boxed{A_n = 0}$

And:

$$\alpha_n B_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r) J_0(\alpha_3 r)\, r\,dr$$

$$= \begin{cases} 0 & \text{if } n \neq 3 \\ 1 & \text{if } n = 3 \end{cases}$$

Here we used orthogonality relations in § 4.8, p252 theorem 1.

Thus:

$$\boxed{u(r,t) = \frac{J_0(\alpha_3 r)}{\alpha_3} \sin(\alpha_3 t)}$$

4.2.10 We would like to solve heat eq on circular regions:

$$\begin{cases} u_t = c^2\left(u_{rr} + \frac{1}{r} u_r\right) & 0 < r < a,\ t > 0 \\ u(a,t) = 0 \\ u(r,0) = f(r) \end{cases}$$

Using separation of variables:

$$u(r,t) = R(r)\,T(t)$$

$$\Rightarrow \quad RT' = c^2\left(R''T + \frac{1}{r} R'T\right)$$

$$\Rightarrow \quad \frac{T'}{c^2 T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2 = \text{const}$$

(choice of sign according to physical principle of exp decaying temp distributions)

We need to solve:

$$(1) \begin{cases} rR'' + R' + \lambda^2 r R = 0 \\ R(a) = 0 \end{cases}$$

and

$$(2) \left\{ T' + \lambda^2 c^2 T = 0 \right.$$

Two linear indep sol to (1) are given by $J_0(\lambda r)$ and $Y_0(\lambda r)$

$$\Rightarrow \quad R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$

But since $Y_0(r) \to -\infty$ as $r \to 0$, we must have $c_2 = 0$

and $R(r) = J_0(\lambda r)$

Using $R(a) = 0$ we get: $\lambda_n a = \alpha_n$, where α_n = n-th zero of J_0.

$$\Rightarrow \quad \boxed{R_n(r) = J_0\left(\frac{\alpha_n}{a} r\right)}$$

Solving (2) we get:

$$T_n(t) = A_n \exp[-\lambda_n^2 c^2 t]$$

Thus by construction $u_n(r,t) = R_n(r) T_n(t)$ solves DE and so does its sum (by linearity)

Thus:

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} A_n \exp\left[-\left(\frac{\alpha_n}{a}\right)^2 c^2 t\right] J_0\left(\frac{\alpha_n}{a} r\right)}$$

To find A_n we use initial cond:

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{a} r\right)$$

$\rightsquigarrow$ A_n are coeff of $f(r)$ in Bessel series expansion

$$\boxed{A_n = \frac{\left(f(r), J_0\left(\frac{\alpha_n}{a} r\right)\right)}{\left(J_0\left(\frac{\alpha_n}{a} r\right), J_0\left(\frac{\alpha_n}{a} r\right)\right)} = \frac{1}{2a^2 J_1^2\left(\frac{\alpha_n}{a} r\right)} \int_0^a r\,dr\, f(r)\, J_0\left(\frac{\alpha_n}{a} r\right)}$$

where we have used the inner prod:

$$(u,v) = \int_0^a r\,dr\, u(r)\, v(r)$$

and the orthogonality relations in §4.8, p 252 theorem 1.