HW5 Solutions
Math 3150-1

$\boxed{3.7.2}$ Solve 2D WEQ with $c = \frac{1}{\pi}$ and

$f(x,y) = \sin \pi x \sin \pi y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$

$g(x,y) = \sin \pi x$

The Fourier series of $f(x,y)$ is itself: $B_{m,n} = \begin{cases} 1 & m=1, n=1 \\ 0 & m \neq 1, n \neq 1 \end{cases}$

This easy to see since

$$B_{m,n} = \frac{\overbrace{(\sin \pi x \sin \pi y, \sin n\pi x \sin m\pi y)}^{\text{2D inner prod}}}{(\sin n\pi x \sin m\pi y, \sin n\pi x \sin m\pi y)} = \underbrace{\frac{(\sin \pi x, \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)}}_{= \begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n \neq 1 \end{cases}} \underbrace{\frac{\overbrace{(\sin \pi y, \sin n\pi y)}^{\text{1D inner prod}}}{(\sin n\pi y, \sin n\pi y)}}_{= \begin{cases} 1 \text{ if } m=1 \\ 0 \text{ if } m \neq 1 \end{cases}}$$

the Fourier coefficients of $g(x,y)$ are $B^*_{m,n} \sqrt{m^2+n^2}$;

thus:

$$B^*_{m,n} \sqrt{m^2+n^2} = \frac{(\sin \pi x, \sin n\pi x \sin m\pi y)}{(\sin n\pi x \sin m\pi y, \sin n\pi x \sin m\pi y)}$$

$$= \underbrace{\frac{(\sin \pi x, \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)}}_{= \begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n \neq 1 \end{cases}} \frac{(1, \sin m\pi y)}{(\sin m\pi y, \sin m\pi y)}$$

Now $(1, \sin m\pi y) = \int_0^1 \sin m\pi y \, dy = -\frac{\cos m\pi y}{m\pi}\Big|_0^1 = \frac{1}{m\pi}(1-(-1)^m)$

$(\sin m\pi y, \sin m\pi y) = \int_0^1 \sin^2 m\pi y \, dy = \frac{1}{2}\int_0^1 1 - \cos 2m\pi y \, dy = \frac{1}{2}$

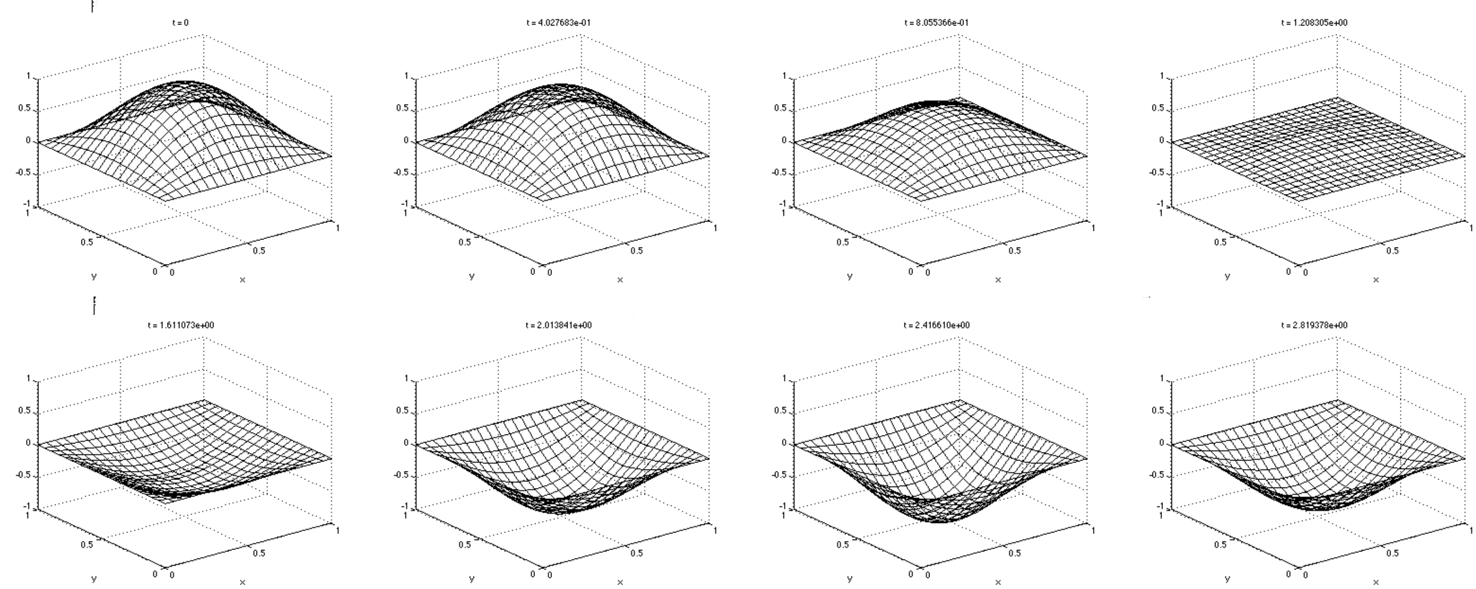
$$\Rightarrow B^*_{m,n} = \begin{cases} \frac{1}{\sqrt{1+m^2}} \frac{1}{2m\pi}(1-(-1)^m) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Putting it all together we ...

$$u(x,y,t) = \sin \pi x \sin \pi y \cos\left(\sqrt{2}\, t\right) + \sum_{m=1}^{\infty} \frac{1}{\sqrt{1+m^2}} \frac{1}{2m\pi} \left(1-(-1)^m\right) \sin \pi x \sin m\pi y \sin\left(\sqrt{1+m^2}\, t\right)$$

(note: I have switched m and n here. but the answer is still correct)

<u>plots:</u>



$\boxed{3.7.13}$ 2D Heat eq. on $[0,1]\times[0,1]$ w/ init temp distrib
$$f(x,y) = \sin\pi x \sin\pi y$$
and $c = 1$.

The Fourier series of $f(x,y)$ is f itself. $\left(A_{n,m} = \begin{cases} 1 & \text{if } n=1, m=1 \\ 0 & \text{otherwise} \end{cases} \right)$

$$\Rightarrow \boxed{u(x,y,t) = \sin\pi x \sin\pi y \exp\left[-\pi\sqrt{2}\, t\right]}$$

$\boxed{3.7.16}$ To prove orthogonality relations in 2D we use these in 1D

Let $(u,v) = \int_0^a \int_0^b u(x,y)\, v(x,y)\, dx\, dy$

then:

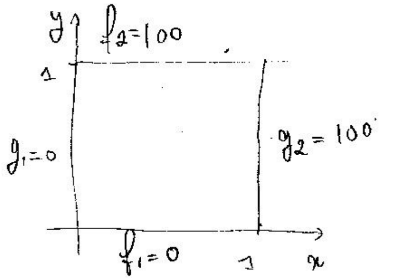
$$\left(\sin\frac{m\pi}{a}x \sin\frac{n\pi y}{b},\ \sin\frac{m'\pi}{a}x \sin\frac{n'\pi y}{b} \right)$$
$$= \left(\sin\frac{m\pi}{a}x,\ \sin\frac{m'\pi}{a}x\right)\left(\sin\frac{n\pi y}{b},\ \sin\frac{n'\pi y}{b}\right)$$
$$= \underbrace{\int_0^a \sin\frac{m\pi}{a}x \sin\frac{m'\pi}{a}x\, dx}_{= \begin{cases} \frac{a}{2} & \text{if } m=m' \\ 0 & \text{otherwise} \end{cases}} \underbrace{\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy}_{= \begin{cases} \frac{b}{2} & \text{if } n=n' \\ 0 & \text{otherwise} \end{cases}}$$

here we used the fact that: $\int_0^a\int_0^b f(x)\,g(y)\,dy\,dx = \left(\int_0^a f(x)dx\right)\left(\int_0^b g(y)dy\right)$

orthogonality relations in 1D. see e.g. p 22

$$= \begin{cases} \frac{ab}{4} & \text{if } m=m' \text{ and } n=n' \\ 0 & \text{otherwise} \end{cases}$$

3-8-2



We need to solve $\Delta u = 0$ with the boundary conditions on the right.

The solution is

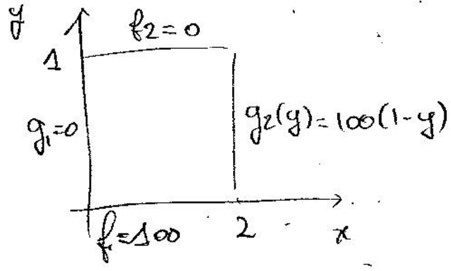
$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(n\pi y) + \sum_{n=1}^{\infty} D_n \sinh(n\pi x)\sin(n\pi y)$$

where $\boxed{B_n} = \frac{2}{\sinh(n\pi)}\int_0^1 100 \sin n\pi x \, dx$

$$= \frac{200}{\sinh(n\pi)} \left. \frac{-\cos n\pi x}{n\pi}\right|_0^1 = \boxed{\frac{200(1-(-1)^n)}{n\pi \sinh(n\pi)}}$$

and $\boxed{D_n = \frac{2}{\sinh n\pi}\int_0^1 100 \sin n\pi y \, dy = B_n}$

3.8.3 We would like to solve:



The solution is:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi}{2}(1-y)\right) + \sum_{n=1}^{\infty} D_n \sin(n\pi y)\sinh(n\pi x).$$

where $\boxed{A_n} = \frac{2}{\sinh(n\pi/2)}\int_0^2 100\sin\left(\frac{n\pi x}{2}\right)dx = \frac{200}{\sinh(n\pi/2)}\left.\left(\frac{-2\cos\frac{n\pi x}{2}}{n\pi}\right)\right|_0^2$

$$\boxed{= \frac{400}{n\pi}\left(1-(-1)^n\right)}$$

and $\boxed{D_n} = \frac{2}{\sinh(2n\pi)}\int_0^1 100(1-y)\sin(n\pi y)\,dy$

$$\overset{IBP}{=} \frac{200}{\sinh(2n\pi)}\left[\left.\frac{(-1+y)\cos(n\pi y)}{n\pi}\right|_0^1 - \int_0^1 \frac{\cos n\pi y}{n\pi}dy\right]$$

$$= \frac{200}{\sinh(2n\pi)}\left[+\frac{1}{n\pi} - \left.\frac{\sin n\pi y}{n\pi}\right|_0^1\right] = \boxed{\frac{200}{n\sinh(2n\pi)}}$$

```
% Math 3150-1
%
% Sample code for problem 3.7.2:
%
% Square membrane with side 1, wave velocity c=1/pi and
%      initial shape      u   (x,y,0) = f(x,y) = sin(pi*x)*sin(pi*y)
%      initial velocity   u_t(x,y,0) = g(x,y) = sin(pi*x)

% number of terms in the expansion
N=30; M=30;

% Time steps and final time
nt=40; Tmax=pi;
ts = linspace(0,Tmax,nt);

% whether to take snapshots for several times
take_snapshots=1;
nsnaps    = 8;
snapbasename = 'p3_7_2';
snapcount = 0;
if (mod(nt,nsnaps)~=0)
 error('number of snapshots must divide number of time steps');
end;

% setup a grid to plot the function
Nx=20; Ny=20; % number of points in x and y directions
x=linspace(0,1,Nx); y=linspace(0,1,Ny);
[xx,yy]=meshgrid(x,y);

for it=1:length(ts),
 t = ts(it);
 ss = sin(pi*xx).*sin(pi*yy)*cos(sqrt(2)*t);
 for n=1:N,
  lambda1n = sqrt(1+n^2);
  B1nstar = (1-(-1)^n)/(lambda1n*2*n*pi);
  ss = ss + sin(pi*xx).*sin(n*pi*yy) * B1nstar * sin(lambda1n*t);
 end;
  % the rest of the loop handles plotting
  mesh(xx,yy,ss,'edgecolor','k','facecolor','none');
  axis([0 1 0 1 -1 1]);
  title(sprintf('t = %d',t)); xlabel('x'); ylabel('y');
  pause(0.2); % 0.2 is the time we pause
  % take a snapshot if necessary
  if (mod(it-1,nt/nsnaps)==0 & take_snapshots)
   filename = sprintf('%s_%03d.png',snapbasename,snapcount);
   fprintf(['saving to file ' filename '\n']);
   print('-dpng','-r50',filename);
   snapcount=snapcount+1;
  end;
end;
```