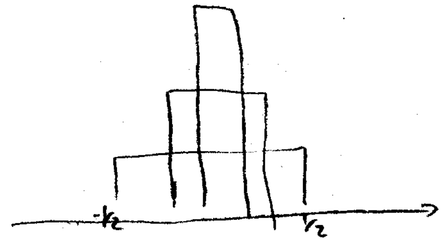
§ 7.8 Generalized functions (distributions)

Consider a force

$$F_k(t) = \begin{cases} k & \text{if } |t| < \frac{1}{2k} \\ 0 & \text{if } |t| > \frac{1}{2k} \end{cases}$$



its impulse is $\displaystyle\int_{-\infty}^{\infty} F_k(t)\, dt = 1$

as $k \to \infty$ this force becomes a force of impulse 1 applied at $t=0$ exactly. However we cannot talk about this force as a conventional function $\to$ distribution or generalized function.

<u>Theorem</u> given f continuous:

$$\lim_{k\to\infty} \int_a^b f(x) F_k(x)\, dx = \begin{cases} f(0) & \text{if } 0 \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

<u>proof</u>: <u>if $0 \notin (a,b)$</u> Then $\exists\, k_0$ s.t. $\forall k \geq k_0$

$$\int_a^b f(x) F_k(x)\, dx = 0$$

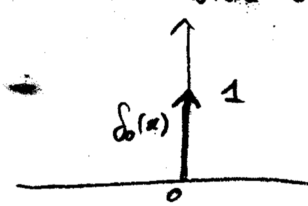
<u>if $0 \in (a,b)$</u> then $\exists\, k_0$ s.t. $\forall k \geq k_0$ $\left(-\frac{1}{2k}, \frac{1}{2k}\right) \subset (a,b)$

$$\Rightarrow \int_a^b f(x) F_k(x)\, dx = k \int_{-\frac{1}{2k}}^{\frac{1}{2k}} f(x)\, dx \underset{\substack{\uparrow \\ \text{MVT}}}{=} k \frac{1}{k} f(c_k)$$

where $c_k \in \left(-\frac{1}{2k}, \frac{1}{2k}\right)$

$$\Rightarrow \lim \int_a^b f(x) F_k(x)\, dx = 0$$

<u>Dirac delta function</u>: $\delta(x) = \delta_0(x)$



$$\int_{-\infty}^{\infty} f(x)\, \delta_0(x)\, dx = f(0)$$

where f is a smooth function supported on a finite interval $[a,b]$.

$$\| \\ \langle \delta_0, f \rangle$$

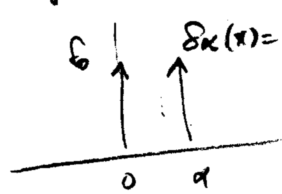
Generalized functions are defined through their action on all test functions within a certain class for ~~ex~~ example all C^∞ functions having derivatives that decay faster than any polynomial at ∞.

Here is another distribution:

$$\boxed{\langle \delta_\alpha , f \rangle = \int_{-\infty}^{\infty} f(x)\, \delta_\alpha(x)\, dx = f(\alpha)}$$

Sometimes: $\delta_\alpha(x) = \delta_0(x-\alpha) = \delta(x-\alpha)$

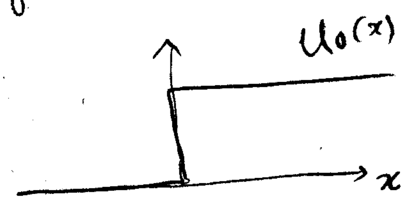
because if δ_α were a function it would be the translate of $\delta_0(x)$.



A usual function is also a generalized function:

Example: Heaviside function

$$U_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



$$\langle U_0, f \rangle = \int_{-\infty}^{\infty} f(x)\, U_0(x)\, dx = \int_0^\infty f(x)\, dx$$

$$U_\alpha(x) = U_0(x-\alpha)$$

Derivatives of generalized functions (IBP)

Let ϕ be a generalized fn..

$$\langle \phi', f \rangle = \int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx = \underbrace{\cancel{f(x)\phi(x)\Big|_{-\infty}^{\infty}}}_{=0} - \int_{-\infty}^{\infty} f'(x)\, \phi(x)\, dx$$

$$= -\langle \phi, f' \rangle = \langle \phi, -f' \rangle$$

The derivative of a generalized function is the gen. fun. ϕ' for which:

$$\boxed{\langle \phi', f \rangle = \langle \phi, -f' \rangle}$$

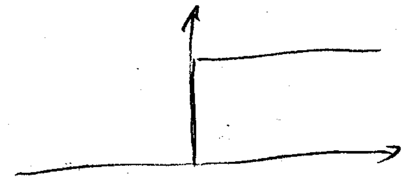
Example: we cannot take derivative of U_0 as a function, but derivative as a gen fun is well defined!

$$\langle U_0', f \rangle = -\langle U_0, f' \rangle = -\int_0^\infty f'(x)\,dx = -f(x)\Big|_0^\infty$$

$$= f(0)$$

$$= \langle \delta_0, f \rangle$$

$\Rightarrow \quad U_0' = \delta_0$



U_0 is constant except at $x=0$ where jump is 1. (infinite slope)

$(c U_\alpha)' = c\,\delta_\alpha$ ← weighted δ function.

function w/ jump at $x=\alpha$ of size c

in general:

$$\boxed{\langle \phi^{(n)}, f \rangle = (-1)^n \langle \phi, f^{(n)} \rangle}$$

(by subsequent applications of integration by parts)

Fourier transform of a gen fun:

$$\langle \hat{\phi}, f \rangle = \langle \phi, \hat{f} \rangle$$

Example:

$$\langle \hat{\delta}_0, f \rangle = \langle \delta_0, \hat{f} \rangle = \int_{-\infty}^{\infty} \hat{f}(x)\,\delta_0(x)\,dx = \hat{f}(0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,dx = \langle \frac{1}{\sqrt{2\pi}}, f \rangle$$

$\Rightarrow \quad \hat{\delta}_0 = \frac{1}{\sqrt{2\pi}}$

Some other Fourier transforms:

$$\mathcal{F}(\delta_\alpha)(\omega) = \frac{1}{\sqrt{2\pi}} e^{-i\alpha\omega}$$

$$\mathcal{F}(\mathcal{U}_0)(\omega) = \frac{-i}{\sqrt{2\pi}\,\omega}$$

$$\mathcal{F}(\mathcal{U}_\alpha)(\omega) = \frac{-i}{\sqrt{2\pi}\,\omega} e^{-i\alpha\omega}$$

how do we get there?

$$\mathcal{F}(\delta_\alpha)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_\alpha(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-i\omega\alpha}$$

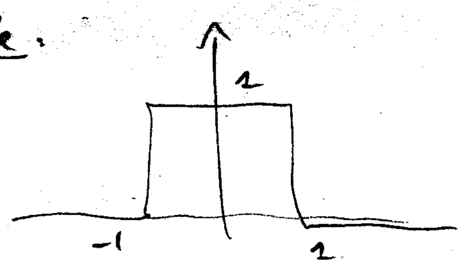
$(\mathcal{U}_\alpha)' = \delta_\alpha \qquad \mathcal{F}((\mathcal{U}_\alpha)'(x))(\omega) = \mathcal{F}(\delta_\alpha)(\omega)$

$$= \frac{1}{\sqrt{2\pi}} e^{-i\alpha\omega}$$

$$= i\omega\, \widehat{\mathcal{U}}_\alpha(\omega)$$

$$\Rightarrow \boxed{\widehat{\mathcal{U}}_\alpha(\omega) = \frac{-ie^{-i\alpha\omega}}{\sqrt{2\pi}\,\omega}}$$

Example.



$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f'(x) = \delta_{-1}(x) - \delta_1(x)$$

$$\widehat{f'}(\omega) = \widehat{\delta_{-1}}(\omega) - \widehat{\delta_1}(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \left(e^{i\omega} - e^{-i\omega}\right) = \sqrt{\frac{2}{\pi}}\, i \sin\omega$$

$$\widehat{f'}(\omega) = i\omega \hat{f}(\omega) \Rightarrow \boxed{\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}}$$

Convolutions:

$$\phi * \psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t)\,\psi(x-t)\,dt$$ (not a precise def of convolution for gen fcn. but enough for our purposes)

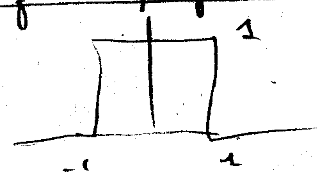
$$\boxed{\delta_\alpha * \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_\alpha(t)\,\phi(x-t) = \frac{1}{\sqrt{2\pi}} \phi(x-\alpha) = \frac{1}{\sqrt{2\pi}} \phi_\alpha(x)}$$

$$(\sqrt{2\pi}\,\delta_0) * f = f$$ $\sqrt{2\pi}\,\delta_0$ acts as identity for convolution operator.

$$(\sqrt{2\pi}\,\delta_\alpha) * f = f(x-\alpha)$$

$$\delta_a * \delta_b = \frac{1}{\sqrt{2\pi}} \delta_{a+b}$$

Convolution of a step function:



$$f * f$$

$$\frac{d^2}{dx^2}(f*f) = f' * f' = (\delta_{-1} - \delta_1) * (\delta_{-1} - \delta_1)$$

$$= \frac{1}{\sqrt{2\pi}} (\delta_{-2} - 2\delta_0 + \delta_2)$$

$$(f*f)' = \frac{1}{\sqrt{2\pi}} (U_{-2} - 2U_0 + U_2)$$

$$f*f = \frac{1}{\sqrt{2\pi}} (\tau_{-2} - 2\tau_0 + \tau_2)$$

where $\tau_\alpha(x) = \int_{-\infty}^{x} U_\alpha(z)\,dz$