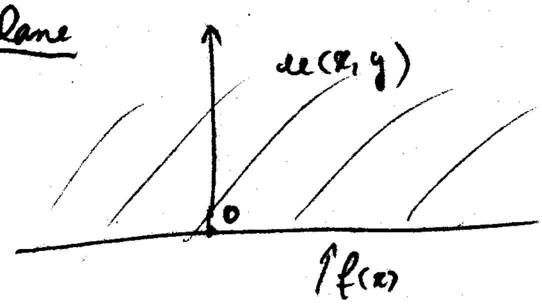
7.5 Dirichlet problem in upper half plane



$$(*)\quad\begin{cases} \Delta u = 0 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

$\downarrow \mathcal{F}$ w.r.t. x

$$\begin{cases} -\omega^2 \hat{u}(\omega, y) + \dfrac{d^2}{dy^2}\hat{u}(\omega, y) = 0 \\ \hat{u}(\omega, 0) = \hat{f}(\omega) \end{cases}$$

$\Rightarrow \hat{u}(\omega, y) = A(\omega)\, e^{-\omega y} + B(\omega) e^{+\omega y}$

but $\hat{u}(\omega, y)$ is bounded Thus $\quad \omega < 0 \Rightarrow A(\omega) = 0$

$\qquad \omega > 0 \Rightarrow B(\omega) = 0$

$\Rightarrow \hat{u}(\omega, y) = C(\omega) e^{-y|\omega|}$

at $y = 0$:

$\hat{u}(\omega, 0) = \hat{f}(\omega) = C(\omega)$

$$\Rightarrow \hat{u}(\omega, y) = \hat{f}(\omega) e^{-y|\omega|} = \hat{f}(\omega)\, \mathcal{F}\Bigg(\underbrace{\sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}}_{\substack{= P_y(x) \\ = \text{Poisson kernel}}}\Bigg)(\omega)$$

(see identity 9. appendix B)

$$\Rightarrow \boxed{u(x, y) = f(x) * P_y(x)}$$

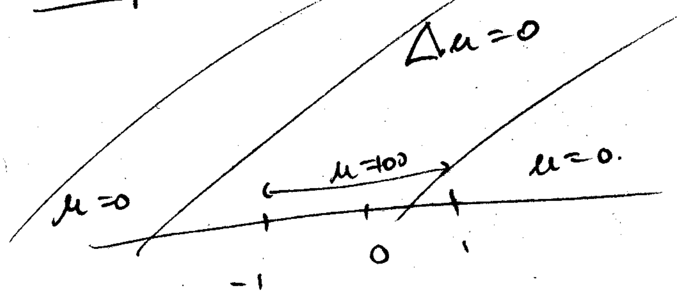
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f(t)\, P_y(x - t)$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt$$

<u>note</u>: problem $(*)$ as stated does not have a unique solution, indeed: if $u(x,y)$ solves $(*)$ then so does $v(x,y) = u(x,y) + ay$ (check), $a \in \mathbb{R}$.

However sol. is unique and equal to what we found if $|u| < M$ (boundedness)

Example 7.5.1



$$\int \frac{dz}{1+z^2} = \text{arc}\tan$$

$$f(x) = \begin{cases} 100 & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$$

$$u(x,y) = \frac{100y}{\pi}\int_{-1}^{1}\frac{dt}{(x-t)^2+y^2} = \frac{100}{\pi y}\int_{-1}^{1}\frac{dt}{1+\left(\frac{x-t}{y}\right)^2}$$

$$= \frac{100}{\pi y}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}}\frac{\cancel{y}\,du}{1+u^2} = \frac{100}{\pi}\left(\operatorname{atan}\frac{1-x}{y} - \operatorname{atan}\frac{-1-x}{y}\right)$$

odd

$u = \frac{t-x}{y}$

$du = \frac{dt}{y}$

$$= \frac{100}{\pi}\left(\operatorname{atan}\frac{1-x}{y} + \operatorname{atan}\frac{1+x}{y}\right)$$

Isotherms: points (x,y) s.t. $u(x,y) = T$ = constant

$$\operatorname{atan}\left(\frac{1-x}{y}\right) + \operatorname{atan}\left(\frac{1+x}{y}\right) = \frac{\pi T}{100}$$

using: $\tan(a+b) = \dfrac{\tan a + \tan b}{1-\tan a \tan b}$

$$\frac{\frac{1-x}{y} + \frac{1+x}{y}}{1 - \frac{(1-x)(1+x)}{y^2}} = \tan\left(\frac{\pi T}{100}\right)$$

$$\parallel$$

$$\frac{2}{y\left(1 - \frac{1-x^2}{y^2}\right)} = \frac{2y}{x^2+y^2-1}$$

thus we get that:

$$x^2 + y^2 - \frac{2y}{\tan\left(\frac{\pi T}{100}\right)} = 1$$

$$x^2 + \left(y - \frac{1}{\tan\left(\frac{\pi T}{100}\right)}\right)^2 = 1 + \frac{1}{\tan^2\left(\frac{\pi T}{100}\right)} = \frac{\sin^2\left(\frac{\pi T}{100}\right) + \cos^2\left(\frac{\pi T}{100}\right)}{\sin^2\left(\frac{\pi T}{100}\right)}$$

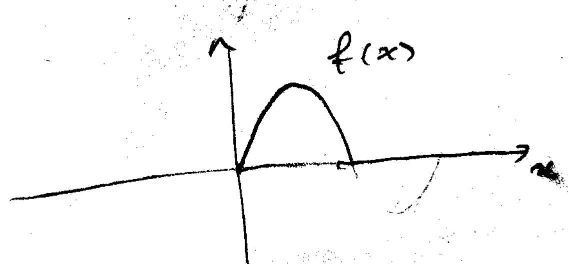
$$= \frac{1}{\sin^2\left(\frac{\pi T}{100}\right)}$$

isotherms are circles of center $\left(0, \frac{1}{\tan\left(\frac{\pi T}{100}\right)}\right)$ and radius $\frac{1}{\sin\left(\frac{\pi T}{100}\right)}$

see plots.

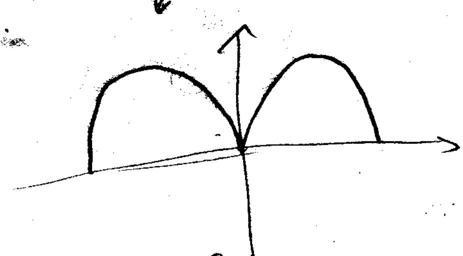
7.6 Fourier Cosine and Sine transforms

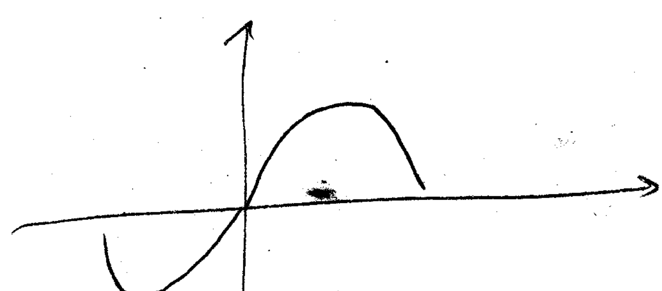
How can we use F.T. for functions that are defined over half line only?



odd extension

even extension





$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt$$

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \sin \omega t \, dt$$

However it is easier to work with:

$$\hat{f}_c(\omega) = \mathcal{F}_c(f)(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt \quad (\omega \geq 0)$$

(<u>Fourier cosine transform</u>)

$$f(x) = \mathcal{F}_c^{-1}(\hat{f}_c(\omega))(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega t \, dt \quad (\omega \geq 0)$$

(Inverse cosine transform)

$$\hat{f}_s(\omega) = \mathcal{F}_s(f)(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt \quad (\omega \geq 0)$$

(<u>Fourier sine transf</u>)

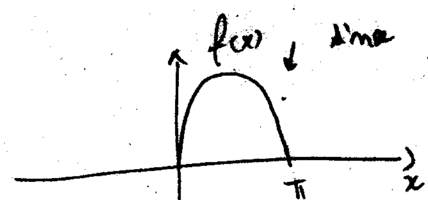
$$f(x) = \mathcal{F}_s^{-1}(\hat{f}_s(\omega))(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) \sin \omega t \, d\omega$$

(<u>Inverse Fourier sine trans</u>)

- the constants in front are the same for transform and inverse transform.
- no sign differences.

Sometimes easier to work with these transforms since they do not involve complex numbers.

<u>Example 7.6.1</u>



$$f(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x \geq \pi \end{cases}$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \cos \omega x \, dx = \cdots = \sqrt{\frac{2}{\pi}} \frac{\cos(\pi\omega) + 1}{1 - \omega^2}$$

using $2 \sin a \cos b = \sin(a-b) + \sin(a+b)$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\pi\omega) + 1}{1 - \omega^2} \cos \omega x \, d\omega$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin \omega x \, dx = \cdots = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi\omega)}{1 - \omega^2}$$

using $2 \sin a \sin b = \cos(a-b) - \cos(a+b)$

$$\rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\pi\omega)}{1 - \omega^2} \sin \omega x \, d\omega$$

As you may have suspected these transforms are closely related to the Fourier transform.

Let f_e = <u>even</u> extension of f , where f defined for $x \geq 0$.
f_o = <u>odd</u> extension of f

then:

$$\boxed{\begin{aligned} \mathcal{F}_c(f)(\omega) &= \mathcal{F}(f_e)(\omega) \\ \mathcal{F}_s(f)(\omega) &= i\,\mathcal{F}(f_o)(\omega) \end{aligned}}$$

this can be easily shown using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$:

For example:

cancels out

stays

$$\mathcal{F}(f_e)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_e(x)\, \underbrace{(\cos\omega x + i\sin\omega x)}_{e^{-i\omega x}}\, dx$$

$$= \underbrace{\frac{2}{\sqrt{2\pi}}}_{=\sqrt{\frac{2}{\pi}}} \int_0^{\infty} f(x) \cos\omega x \, dx = \mathcal{F}_c(f)(\omega)$$

and similarly for $\mathcal{F}_s$ ...

working with the {cosine, sine} Fourier transforms is very similar to working with the Fourier transform:

$$\boxed{\begin{aligned} \mathcal{F}_c(\alpha f+\beta g) &= \alpha \mathcal{F}_c(f) + \beta \mathcal{F}_c(g) \\ \mathcal{F}_s(\alpha f+\beta g) &= \alpha \mathcal{F}_s(f) + \beta \mathcal{F}_s(g) \end{aligned}}$$ (linearity)

Assuming $f(x) \to 0$ as $x \to \infty$

$$\boxed{\begin{aligned} \mathcal{F}_c(f') &= \omega \mathcal{F}_s(f) - \sqrt{\tfrac{2}{\pi}}\, f(0) \\ \mathcal{F}_s(f') &= -\omega \mathcal{F}_c(f) \end{aligned}}$$

$$\boxed{\begin{aligned} \mathcal{F}_c(f'') &= -\omega^2 \mathcal{F}_c(f) - \sqrt{\tfrac{2}{\pi}}\, f'(0) \\ \mathcal{F}_s(f'') &= -\omega^2 \mathcal{F}_s(f) + \sqrt{\tfrac{2}{\pi}}\, \omega f(0) \end{aligned}}$$ (if $f(x) \to 0$ as $x \to \infty$, $f'(x) \to 0$)

other properties:

$$\boxed{\begin{aligned} \mathcal{F}_c(x f(x)) &= \frac{d}{d\omega} \mathcal{F}_s(f(x)) \\ \mathcal{F}_s(x f(x)) &= -\frac{d}{d\omega}\left(\mathcal{F}_c(f(x))\right) \end{aligned}}$$

§7.7 Problems involving semi-infinite intervals

Example 7.7:1

$$\begin{cases} u_t = c^2 u_{xx} & , x>0,\ t>0 \\ u(x,0) = f(x) & x>0 \\ u(0,t) = 0 & t>0 \end{cases}$$

problem is defined only over half line. What transform $\mathcal{F}_s$, $\mathcal{F}_c$ to use?

$\mathcal{F}_c$: we do not have u_x available to compute $\mathcal{F}_c(u_{xx})$ not good!

$\mathcal{F}_s$:

$$\mathcal{F}_s(u_t) = c^2 \mathcal{F}_s(u_{xx})$$

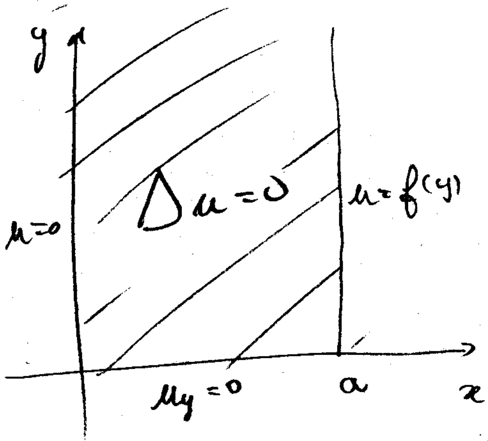
$$\begin{cases} \frac{d}{dt}\hat{u}_s(\omega,t) = c^2\left(-\omega^2 \hat{u}_s(\omega,t) + \cancel{\sqrt{\frac{2}{\pi}}\,\omega\, u(0,t)}^{=0}\right) \\ \hat{u}_s(\omega,0) = \hat{f}_s(\omega) \end{cases}$$

solving for $\hat{u}_s(\omega,t) = A(\omega)\, e^{-c^2\omega^2 t}$

using I.C. $\boxed{\hat{u}_s(\omega,t) = \hat{f}_s(\omega)\, e^{-c^2\omega^2 t}}$

$$\boxed{u(x,t) = \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_s(\omega,t) \sin \omega x \, d\omega}_{\text{inverse Sine transform}} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega)\, e^{-c^2\omega^2 t} \sin \omega x \, d\omega}$$

Example 7.7.2 Dirichlet-Neumann problem on semi infinite strip



$$\begin{cases} \Delta u = 0 & 0 < x < a,\ y > 0 \\ u_y(x,0) = 0, & 0 < x < a \\ u(0,y) = 0, & u(a,y) = f(y) \end{cases}$$

what transform to use? w.r.t to which variables?

- transform w.r.t. y variables because they are semi infinite var.
- $\mathcal{F}_s$ requires $u(x,0)$ for computing $\mathcal{F}_s(u(x,y))(\omega)$ ⇝ we don't have it.

$$\begin{cases} \mathcal{F}_c(\Delta u) = \frac{\partial^2}{\partial x^2} \hat{u}_c(x,\omega) - \omega^2 \hat{u}_c(x,\omega) - \cancel{\sqrt{\frac{2}{\pi}}\, \underbrace{u_y(x,0)}_{=0}} = 0 \\ \hat{u}_c(0,\omega) = 0 \quad ; \quad \hat{u}_c(a,\omega) = \hat{f}_c(\omega) \end{cases}$$

$$\Rightarrow \hat{u}_c(x,\omega) = A(\omega)\cosh \omega x + B(\omega)\sinh \omega x$$

$$\hat{u}_c(0,\omega) = A(\omega) = 0$$

$$\hat{u}_c(a,\omega) = B(\omega)\sinh \omega a = \hat{f}_c(\omega)$$

$$\rightarrow \hat{u}_c(x,\omega) = \frac{\hat{f}_c(\omega)}{\sinh \omega a} \sinh \omega x$$

$$\Rightarrow \boxed{u(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_c(x,\omega) \cos \omega y \, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\hat{f}_c(\omega)}{\sinh \omega a} \sinh \omega x \cos \omega y \, d\omega}$$