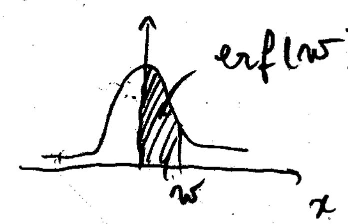
In other words we can write:

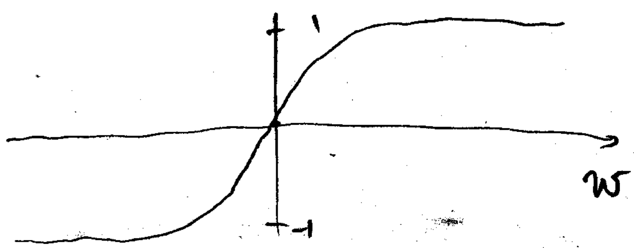
$$u(x,t) = (g_t * f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)\, g_t(s)\, ds$$

$$= (f * g_t)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, g_t(x-y)\, dy$$

We will need to know integrals of the Gaussian functions, these are tabulated and available in Matlab, Maple, Mathematica etc...

$$\operatorname{erf}(w) = \frac{2}{\sqrt{\pi}}\int_0^{w} e^{-z^2}\, dz$$





Example 7.4.1 Solve heat eq with $f(x) = \begin{cases} 100 & \text{if } |x|<1 \\ 0 & \text{if } |x|>0 \end{cases}$

and $c=1$

$$\Rightarrow \quad u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 100\,\frac{1}{\sqrt{2t}}\, e^{-(x-s)^2/4t}\, ds$$

C.O.V

$z = (x-s)/2\sqrt{t}$

$dz = -ds/2\sqrt{t}$

$$= \frac{100}{2\sqrt{\pi t}}\int_{\frac{x+1}{2\sqrt{t}}}^{\frac{x-1}{2\sqrt{t}}} e^{-z^2}\left(-2\sqrt{t}\, dz\right)$$

$$= \frac{100}{\sqrt{\pi}}\int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-z^2}\, dz = \frac{100}{\sqrt{\pi}}\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right)\right]$$

show animation.

Here is an example that is easier to do with the Fourier method

Example 7.4.2 Heat eq with nonconstant coefficients

the "c" depends on time.

$$\begin{cases} u_t = t\, u_{xx} \\ u(x,0) = f(x) \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} \frac{d}{dt}\hat{u}(\omega;t) = -t\omega^2 \hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\hat{u}(\omega,t) = \hat{f}(\omega) e^{-\frac{t^2}{2}\omega^2}$$

Gaussian

$$\Rightarrow u(x,t) = \left(f * \mathcal{F}^{-1}\left(e^{-\frac{t^2}{2}\omega^2}\right)\right)(x)$$

actually: $\mathcal{F}\left(\frac{e^{-x^2/2t^2}}{t}\right)(\omega) = e^{-\frac{t^2\omega^2}{2}}$

$$\Rightarrow u(x,t) = \left(f * \frac{e^{-\frac{x^2}{2t^2}}}{t}\right)(x) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-s)^2}{2t^2}}\, ds.$$

for example $f(x) = \begin{cases} 100 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

$z = \frac{x-s}{\sqrt{2}\,t};\ dz = -\frac{ds}{\sqrt{2}\,t}$

$$u(x,t) = \frac{100}{t\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{2t^2}}\, ds$$

$$= \frac{100}{t\sqrt{2\pi}} \int_{\frac{x+1}{\sqrt{2}t}}^{\frac{x-1}{\sqrt{2}t}} e^{-z^2}(-\sqrt{2}\,t)\, dz$$

$$= \frac{100}{\sqrt{\pi}} \left(\int_0^{\frac{x+1}{\sqrt{2}t}} e^{-z^2} dz - \int_0^{\frac{x-1}{\sqrt{2}t}} e^{-z^2} dz \right)$$

$$= 50\left[\operatorname{erf}\left(\frac{x+1}{\sqrt{2}\,t}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{2}\,t}\right)\right]$$