§7.3 The Fourier Transform method

Let $u(x,t)$ be a smooth function def for $x \in \mathbb{R}, t>0$.

The F.T. in the x-variable is:

$$\mathcal{F}(u(x,t))(\omega) = \hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$$

F.T. of partial derivatives.

$$\mathcal{F}\left(\frac{\partial^n}{\partial t^n} u(x,t)\right)(\omega) = \frac{d^n}{dt^n} \hat{u}(\omega,t), \quad n=1,2,\ldots$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial x^n} u(x,t)\right)(\omega) = (i\omega)^n\, \hat{u}(\omega,t)$$

comes from interchanging $\frac{\partial}{\partial t}$ and integral.

$$\frac{d}{dt}\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t)\, e^{-i\omega x}\, dx$$

Example 7.3.1 The wave equation for an infinite string

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty,\ t>0 \\ u(x,0) = f(x) & \text{(initial shape)} \\ u_t(x,0) = g(x) & \text{(initial velocity)} \end{cases}$$

F.T.

$$\begin{cases} \frac{d^2}{dt^2} \hat{u}(\omega,t) = -c^2\omega^2 \hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \\ \frac{d}{dt}\hat{u}(\omega,0) = \hat{g}(\omega) \end{cases}$$

General sol to this ODE in t is:

$\Rightarrow \hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t.$

(ω goes free for the ride, but since we have one ODE to solve for each frequency the constants might depend on ω)

Use B.C.

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\hat{u}_t(\omega,t) = -c\omega A(\omega)\sin c\omega t + c\omega B(\omega)\cos c\omega t$$

$$\hat{u}_t(\omega,0) = c\omega B(\omega) = \hat{g}(\omega)$$

thus:

$$\hat{u}(\omega,t) = \hat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\sin c\omega t$$

$\rightarrow$ go back to space domain:

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega,t))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\sin c\omega t \times e^{+i\omega x}\, d\omega.$$

PDE in x,t $\xrightarrow{\mathcal{F}}$ ODE in ω,t $\Rightarrow$ sol in ω/t $\xrightarrow{\mathcal{F}^{-1}}$ sol in x,t.

Example 7.3.2: The heat eq on an infinite rod

$$\begin{cases} u_t = c^2 u_{xx} & -\infty < x < \infty,\ t>0 \\ u(x,0) = f(x) & \text{(initial temp distrib)} \end{cases}$$

$\downarrow \mathcal{F}$

$$\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,t) = \hat{f}(\omega) \end{cases}$$

The general solution to this ODE in t is of the form:

$$\hat{u}(\omega,t) = A(\omega)\, e^{-c^2\omega^2 t}$$

Using I.C.:

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$$

$\mathcal{F}^{-1}$

$$u(x,t) = \left(\mathcal{F}^{-1}\hat{u}(\omega,t)\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega$$

$=$ convolution (next section)

<u>Example 7.3.3</u>

$$\begin{cases} u_{tx} = u_{xx} \\ u(x,0) = \sqrt{\frac{\pi}{2}}\, e^{-|x|} \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} i\omega \dfrac{d\hat{u}}{dt}(\omega,t) = -\omega^2 \hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, e^{-|x|}\right) = \dfrac{1}{1+\omega^2} \end{cases}$$

$$\Rightarrow \hat{u}(\omega,t) = A(\omega)\, e^{i\omega t}$$

$$\hat{u}(\omega,0) = A(\omega) = \frac{1}{1+\omega^2} \Rightarrow \hat{u}(\omega,t) = \frac{1}{1+\omega^2} e^{i\omega t}$$

use shifting property (hw problem):

$$\mathcal{F}^{-1}\left(e^{i\omega t}\frac{1}{1+\omega^2}\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{1+\omega^2}\right)(x+t)$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-|x+t|}$$

Example 7-3-5:

$$\begin{cases} t\,u_x + u_t = 0 \\ u(x,0) = f(x) \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} i\omega t\,\hat{u}(\omega,t) + \frac{d}{dt}\hat{u}(\omega,t) = 0 \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\hat{u}(\omega,t) = A(\omega)\, e^{-i\frac{t^2}{2}\omega}$$

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-i\frac{t^2}{2}\omega}$$

$$\Rightarrow \hat{u}(x,t) = f\left(x - \frac{t^2}{2}\right)$$

7.4 The Heat Equation and Gauss's kernel

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}$$

recall from previous section that:

$$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$$

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega,t))(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, dz$$

Now notice that

$$\left(\mathcal{F}^{-1} e^{-c^2\omega^2 t}\right)(x) = \frac{1}{c\sqrt{2t}}\, e^{-x^2/4c^2t} = \text{heat kernel} \mid \text{Gauss kernel}$$

$$= g_t(x)$$

note: called kernel because it can be used to generate sol to heat eq with any rhs.

$$\Rightarrow \hat{u}(\omega,t) = \hat{g}_t(\omega)\, \hat{f}(\omega)$$

$$\Rightarrow \boxed{u(x,t) = (g_t * f)(x)}$$