Separation of variables $u(r,z) = R(r)\,Z(z)$

$$(1)\begin{cases} r^2R''+rR'-kr^2R=0 \\ R(a)=0\end{cases} \qquad (2)\begin{cases} Z''+kZ=0 \\ Z(0)=0\end{cases}$$

Solving (1) (slightly more complicated than circular case)

$\underline{k=0}$

$$\begin{cases} rR''+R'=0 & \Rightarrow R(r)=\alpha\ln r \\ R(a)=0 & \Rightarrow R(r)=0. \end{cases}$$

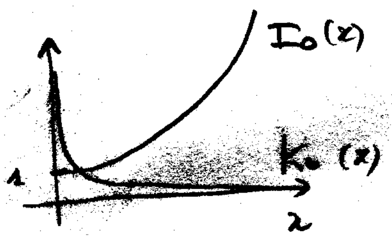
blows up at $r=0$

$\underline{k=-\lambda^2<0}$ sol to (1) is $R(r)=\alpha\,\underbrace{J_0(\lambda r)+\beta\,Y_0(\lambda r)}$

$R(a)=0 \Rightarrow \beta=0.$

$\Rightarrow \boxed{R_n(r)=J_0\left(\frac{\alpha_n}{a}r\right)}$, $\alpha_n=$ n-th zero of $J_0(\cdot)$

$\underline{k=\lambda^2>0}$ sol to (1) is $R(r)=\alpha I_0(\lambda r)+\beta K_0(\lambda r)$



~~K_0 is unbounded~~ at $r=0$ thus $R(a)=0 \Rightarrow \beta=0$

I_0 grows and $I_0(0)=1 \rightarrow \alpha=0.$

$I_0=$ modified Bessel function of the first kind

$K_0=$ " " " " second kind

$$i^p I_p(x)=J_p(ix) \qquad (\sim \sin ix = i\sinh x)$$

$$K_p(x)=\frac{\pi}{2\sin p\pi}\left[I_{-p}(x)-I_p(x)\right] \quad (\sim \text{same way } Y \text{ was obtained from } J)$$

$p=0$ case obtained as $p\to 0$ above.

Bessel functions J_p, Y_p solve:

$$r^2y''+ry'+(r^2-p^2)y=0$$

Modified Bessel func I_p, K_p solve

$$r^2y''+ry'+(r^2+p^2)y$$

do change of vars: $r=ix$

$$\frac{\partial}{\partial r}=\frac{\partial x}{\partial r}\frac{\partial}{\partial x}=-i\frac{\partial}{\partial x}$$

$$(ix)^2(-i)^2\frac{\partial^2y}{\partial x^2}+(ix)(-i)\frac{\partial y}{\partial x}+((ix)^2-p^2)\,y=0$$

Thus only case $k = -\lambda^2$ is interesting

$$Z_n(z) = \alpha \cosh \lambda_n z + \beta \sinh \lambda_n z$$

$$Z_n(0) = \alpha = 0 \Rightarrow \boxed{Z_n(z) = \sinh \lambda_n z}$$

Thus solution is of the form:

$$\boxed{u(r,z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \sinh \lambda_n z}$$

B.C.:

$$u(r,h) = f(r) = \sum_{n=1}^{\infty} \underbrace{A_n \sinh(\lambda_n h)}_{\text{Bessel series coeff of } f(r)} J_0(\lambda_n r)$$

$$\Rightarrow \boxed{A_n = \frac{1}{\sinh(\lambda_n h)} \frac{2}{a^2 J_1(\alpha_n)} \int_0^r J_0(\lambda_n r) f(r) r\,dr}$$

Dirichlet problem with non zero ~~lateral~~ ~~bdry~~

Very similar derivation, however case $k = \lambda^2$ is the interesting one

$$\begin{cases} rR'' + R' - kr^2 R = 0 \\ R(a) = 1 \end{cases} \qquad \begin{cases} Z'' + kZ = 0 \\ Z(0) = Z(h) = 0 \end{cases}$$

$$\Rightarrow R_n(r) = I_0\left(\frac{n\pi}{h} r\right) \qquad \Rightarrow Z_n(z) = \sin \frac{n\pi}{h} z$$

$$\Rightarrow u(r,z) = \sum_{n=1}^{\infty} B_n I_0\left(\frac{n\pi}{h} r\right) \sin \frac{n\pi}{h} z$$

B.C.

$$f(z) = u(a,z) = \sum_{n=1}^{\infty} \underbrace{B_n I_0\left(\frac{n\pi}{h} a\right)}_{\text{coeff in sine series}} \sin \frac{n\pi}{h} z.$$

$$\Rightarrow B_n = \frac{1}{I_0\left(\frac{n\pi a}{h}\right)} \frac{2}{h} \int_0^h f(z) \sin \frac{n\pi}{h} z\, dz$$

§ 7 The Fourier transform and its applications

Recall:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$$

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt.$$

Thm Fourier integral representation

Let f be piecewise smooth on every finite interval and

$$\int_{-\infty}^{\infty} |f(x)|\,dx < +\infty \quad (\text{"}f \text{ is integrable})$$

Then

$$f(x) = \int_0^{\infty} \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega \quad (x \in \mathbb{R}) \quad (*)$$

where for $\omega > 0$:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

The integral $(*)$ converges to $f(x)$ if f is cont at x

$\frac{f(x+)+f(x-)}{2}$ if f is discont at x.

We are not going to prove this but Fourier series ~ Rieman Sum for $(*)$

when $p \gg 1$:

$$a_n \approx \frac{1}{p}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi}{p}t\,dt = A(\omega_n)\Delta\omega$$

where $\omega_n = \frac{n\pi}{p}$, $\Delta\omega = \frac{\pi}{p}$.

$$b_n \approx B(\omega_n)\Delta\omega$$

thus: $\boxed{f(x) \approx \sum_{n=1}^{\infty} \left(A(\omega_n) \cos \omega_n x + B(\omega_n) \sin \omega_n x \right) \Delta\omega}$

Example:

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt = \frac{1}{\pi} \int_{-1}^{1} \cos \omega t \, dt = \frac{2 \sin \omega}{\pi \omega}$$

$B(\omega) = 0$ (evenness)

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \cos \omega x}{\omega} \, d\omega = \begin{cases} 1 & \text{if } |x| < 1 \\ \frac{1}{2} & \text{if } |x| = 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

§7.2 The Fourier transform

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

Recall Fourier integral representation:

$$f(x) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} dt\, f(t)\,(\cos\omega t \cos\omega x + \sin\omega t \sin\omega x)$$

$$= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} dt\, f(t)\, \cos\omega(x-t)$$

$\downarrow$ Euler's formula

$$= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} dt\, f(t)\, \frac{e^{i\omega(x-t)} + e^{-i\omega(x-t)}}{2},$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega(x-t)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x} \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t)}_{= \hat{f}(\omega) = (\mathcal{F} f)(\omega)}$$

Fourier transform

$$\hat{f}(\omega) = (\mathcal{F}f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t)$$

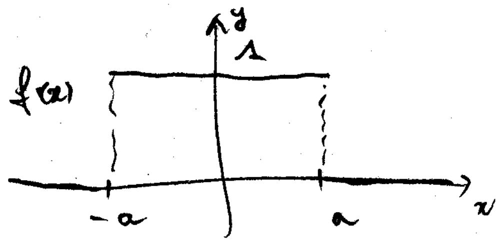
Inverse Fourier transform

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x} \hat{f}(\omega)$$

$$\boxed{f = \mathcal{F}^{-1}\mathcal{F}f}$$

Note: the $\frac{1}{\sqrt{2\pi}}$ factor is designed so that both Fourier and inverse Fourier transform have the same factor. Other conventions are possible.

Example:

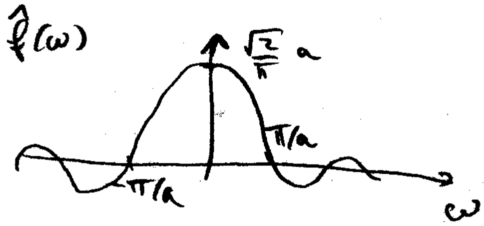


$$f(x) = \begin{cases} 1 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

$$\boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-i\omega x} f(x)} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-i\omega} e^{-i\omega x} \Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{i} (e^{-i\omega a} - e^{+i\omega a}) = \boxed{\frac{\sqrt{2}}{\pi} \frac{\sin a\omega}{\omega}}$$



What happens at $\omega = 0$? Think of Taylor:

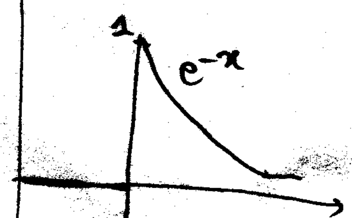
$$\sin a\omega = a\omega + o(\omega)$$

$$\Rightarrow \lim_{\omega \to 0} \frac{\sin a\omega}{\omega} = a$$

Sine cardinal (sinc) function. Important in signal processing, optics...

$$\Rightarrow \boxed{\hat{f}(0) = \frac{\sqrt{2}}{\pi} a}$$

Example



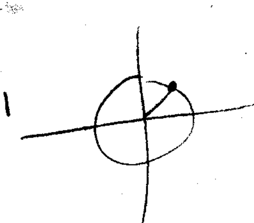
$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-i\omega x} e^{-x} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(i\omega + 1)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{(i\omega + 1)} e^{-x(i\omega+1)} \Big|_0^{\infty}$$

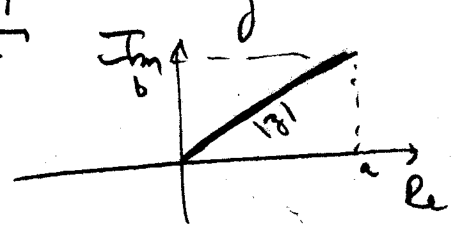
Since $|e^{-x(i\omega+1)}| = |e^{-i\omega x}|\, e^{-x} = e^{-x}$

we have $\lim_{x \to \infty} e^{-x(i\omega+1)} = 0.$ $\quad |e^{i\theta}| = 1$



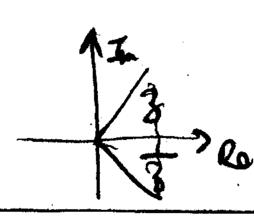
Here $|z|$ is modulus or magnitude of a complex number $z = a + ib$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$$



$$|z|^2 = z\bar{z} = (a+ib)(a-ib)$$

$$\bar{z} = a - ib = \text{conjugate of } z$$



thus
$$\boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}(1+i\omega)} = \frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)}}$$

Note: it is usually better to put any imaginary numbers in numerator.
recall that if $z = a+ib$ $(a, b \in \mathbb{R})$:
$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{|z|^2}\bar{z}$$

Note: Fourier transform of real function is in general complex

Fourier transform calculus

Fourier transform takes differential eq and transforms it into algebraic eq: easier to handle.

Linearity: $\mathcal{F}(\alpha f + \beta g) = \alpha(\mathcal{F} f) + \beta(\mathcal{F} g)$

Derivatives:
- $\mathcal{F}(f') = i\omega(\mathcal{F} f)$ (f, f' integrable, $f(x) \to 0$ as $|x| \to \infty$)
- $\mathcal{F}(f'') = (i\omega)^2(\mathcal{F} f) = -\omega^2(\mathcal{F} f)^2$ (f, f', f'' integrable, $f, f' \to 0$ as $|x| \to \infty$)
- $\mathcal{F}(f^{(n)}) = (i\omega)^n \mathcal{F} f$ ($f, f', \dots, f^{(n)}$ integrable, $f, f', \dots f^{(n-1)} \to 0$ as $|x| \to \infty$)

Proof: IBP.
$$\boxed{(\mathcal{F} f')(\omega)} = \int_{-\infty}^{\infty} dx\, e^{-i\omega x} f'(x) = \underbrace{e^{-i\omega x} f(x)\Big|_{-\infty}^{\infty}}_{=0\ (|e^{-i\omega x}|=1)} + (i\omega)\int_{-\infty}^{\infty} dx\, e^{-i\omega x} f(x)$$
$$\boxed{= (i\omega)(\mathcal{F} f)(\omega)}$$

other relations obtained by repeated application of first one

Derivatives of Fourier transforms

this step needs to be justified more...

$$\boxed{(\hat{f})'(\omega)} = \frac{d}{d\omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{d}{d\omega}e^{-i\omega x}\,dx$$

$$= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x f(x) e^{-i\omega x}\,dx = \boxed{\frac{1}{i}\mathcal{F}(x f(x))(\omega)}$$

in general:

$$\boxed{\mathcal{F}(x^n f(x))(\omega) = i^n (\hat{f})^{(n)}(\omega)}$$

Convolution

convenient factor, maybe defined differently in other books.

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt$$

Easy to show that $(f*g)(x) = (g*f)(x)$ (HW problem)

Fourier transform of convolutions

$$\mathcal{F}(f*g) = \mathcal{F}(f)\,\mathcal{F}(g) \qquad (\widehat{f*g} = \hat{f}\,\hat{g})$$

proof: $$\mathcal{F}(f*g)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i\omega x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, f(x-t)\,g(t)$$

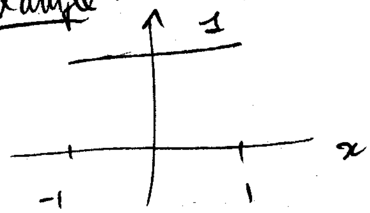
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, f(x-t)\,e^{-i\omega x} g(t)$$

COV. $u = x-t$, $du = dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, f(u)e^{-i\omega u}\, g(t)\,e^{-i\omega t}$$

$$= (\mathcal{F}f)(\omega)\,(\mathcal{F}g)(\omega)$$

Example:



$$f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$$\hat{f}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$\mathcal{F}(f * f)(x) = \left(\hat{f}(\omega)\right)^2 = \frac{2}{\pi}\,\frac{\sin^2\omega}{\omega^2}$$

$$(f * f)(x) = \mathcal{F}^{-1}\left(\frac{2}{\pi}\,\frac{\sin^2\omega}{\omega^2}\right)(x)$$

$$\underset{\text{Formula 11, Appendix B}}{=} \begin{cases} \sqrt{\frac{2}{\pi}}\left(1 - \frac{|x|}{2}\right) & \text{if } |x| < 2 \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Gaussian function

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

To prove this identity we use a sneaky trick:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r\,dr\, e^{-r^2} = (2\pi)\left(-\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\right) = \pi$$

Fourier transform of Gaussian function

For $a > 0$:

$$\boxed{\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}}\, e^{-\frac{\omega^2}{2a}}}$$

($a = 1$: $\widehat{e^{-\frac{x^2}{2}}}(\omega) = e^{-\frac{\omega^2}{2}}$, Gaussian Fourier transform is function itself) proof: (nice proof in book)

Let $f(x) = e^{-\frac{ax^2}{2}}$ then:

$$f' + a x f(x) = 0$$

$$\mathcal{F} \downarrow$$

$$\omega \hat{f}(\omega) + a\left(\hat{f}(\omega)\right)' = 0$$

$$\Rightarrow \quad \hat{f}(\omega) = A e^{-\frac{\omega^2}{2a}} \qquad u = \sqrt{\tfrac{a}{2}}\, x$$

$$\hat{f}(0) = A = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{a\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{a}}.$$