Now $c_2 = 0$ since we do not want unbounded solutions for $r=0$.

thus:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right]$$

$$\Rightarrow u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

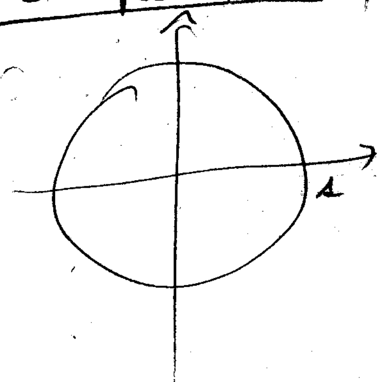
$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

Example 4.4.1

Steady state temp distrib on disk of radius 1 with $u(1,\theta) = f(\theta) = \begin{cases} 100 & 0<\theta<\pi \\ 0 & \pi<\theta<2\pi \end{cases}$



$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 100\, d\theta = 50$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 100 \cos n\theta \, d\theta = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 100 \sin n\theta \, d\theta = -\frac{100}{n\pi} \cos n\theta \Big|_0^{\pi}$$

$$= \frac{100}{n\pi}(1-\cos n\pi)$$

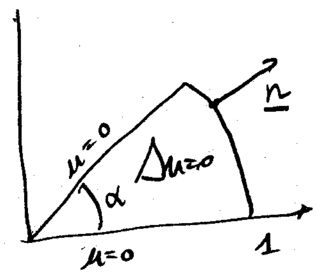
$$\Rightarrow u(r,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(1-\cos n\pi)\, r^n \sin n\theta$$

Of course by construction at $r=1$ we should get back Fourier series of f.

Other regions and Boundary conditions

With polar coordinates it is relatively easy to solve Laplace equation on annulus, wedges, or outside disk.

Example 4.4.4 Laplace equation on a wedge with Robin B.C.



$$\begin{cases} \Delta u = 0 & \text{on wedge} \\ u = 0 & \text{on sides of wedge} \\ \underline{n}\cdot\nabla u + u = -\theta & \text{on } r=1 \text{ boundary of wedge} \end{cases}$$

(Recall Robin B.C. are in general:
$\underline{n}\cdot\nabla u + a u = f$
$\underline{n}$ = unit outward pointing vector normal to boundary)

In our case $\underline{n}\cdot\nabla u\Big|_{r=1} = \dfrac{\partial u}{\partial n}\Big|_{r=1} = \dfrac{\partial u}{\partial r}(1,\theta)$

Thus B.C. on part of wedge where $r=1$ is:

$$u_r(1,\theta) + u(1,\theta) + \theta = 0$$

<u>Method</u>: separation of variables

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

as before we get (1) $\begin{cases} \Theta'' + \lambda^2\Theta = 0 \\ \Theta(0) = \Theta(\alpha) = 0 \end{cases}$

and (2) $r^2R'' + rR' - \lambda^2 R = 0$

Solving (1) we get $\Theta(\theta) = a\cos\lambda\theta + b\sin\lambda\theta$, but $\Theta(0)=0 \Rightarrow a=0$ and $\Theta(\alpha)=0 \Rightarrow \lambda = \lambda_n = \dfrac{n\pi}{\alpha}$

$$\Rightarrow \boxed{\Theta_n = \sin\frac{n\pi}{\alpha}\theta} \qquad n=1,2,\dots$$

Now we get for (2):

$$r^2R'' + rR' - \left(\frac{n\pi}{\alpha}\right)^2 R = 0$$

Solution is of the form $R(r) = a\, r^{\frac{n\pi}{\alpha}} + b r^{-\frac{n\pi}{\alpha}}$

But since $R(0)$ must be bounded we have $b = 0$

$$\Rightarrow \quad R_n(r) = r^{\frac{n\pi}{\alpha}}$$

and $u_n(r,\theta) = b_n\, r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right)$

solves DE by construction.

$\Rightarrow$ Solution is of the form:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} b_n\, r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right)}, \quad r \in [0,1], \ \theta \in [0,\alpha].$$

To find b_n we look at B.C.:

$$u_r(r,\theta) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{\alpha} r^{\left(\frac{n\pi}{\alpha}-1\right)} \sin\left(\frac{n\pi}{\alpha}\theta\right)$$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{\alpha} \sin \frac{n\pi}{\alpha}\theta$$

$$\underset{\substack{\uparrow \\ \text{Robin B.C.}}}{=} -u(1,\theta) - \theta$$

$$= -\theta - \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\alpha}\theta$$

$$\Rightarrow \quad -\theta = \sum_{n=1}^{\infty} \underbrace{b_n\left(1 + \frac{n\pi}{\alpha}\right)} \sin \frac{n\pi}{\alpha}\theta$$

Coeff in half range sine series of $-\theta$:

$$b_n\left(1 + \frac{n\pi}{\alpha}\right) = -\frac{2}{\alpha}\int_0^{\alpha} \theta \sin \frac{n\pi}{\alpha}\theta \, d\theta$$

$$= -\frac{2}{\alpha}\left[-\theta \frac{\alpha}{n\pi}\cos \frac{n\pi}{\alpha}\theta \Big|_0^{\alpha} + \frac{\alpha}{n\pi}\int_0^{\alpha} \cos \frac{n\pi}{\alpha}\theta \, d\theta\right]$$

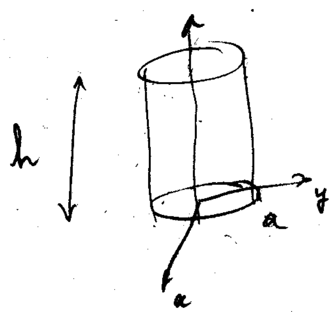
$$= \frac{2\alpha(-1)^n}{n\pi}$$

To conclude:

$$\boxed{u(r,\theta) = \frac{2\alpha^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(\alpha + n\pi)} r^{\frac{n\pi}{\alpha}} \sin \frac{n\pi}{\alpha}\theta.}$$

§ 4.5 Laplace Equation in a Cylinder

Recall cylindrical coordinates (~ polar in xy plane + height z)



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\boxed{\Delta} u = u_{xx} + u_{yy} + u_{zz}$$

$$= \underbrace{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}}_{\text{Laplacian in polar coord.}} + u_{zz}$$

The domain we work on is a cylinder defined by:

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $0 \leq z \leq h$.

Consider first the case where:

- bottom of cylinder is kept at temp 0
- lateral sides of cylinder
- top part of cylinder has a radially symmetric temperature

Then the PDE we would like to solve is:

$$\begin{cases} \Delta u = 0 & \text{inside cylinder} \\ u(r,0) = 0 & 0 \leq r \leq a \\ u(a,z) = 0 & 0 \leq z \leq h \\ u(r,h) = f(r) & 0 \leq r \leq a \end{cases}$$