- Initial velocity gives:

$$u_t(r,0) = g(r) = \sum_{n=1}^{\infty} c\lambda_n B_n J_0(\lambda_n r)$$

$$\Rightarrow \quad c\lambda_n B_n = \frac{(g(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))}$$

$$B_n = \frac{1}{c\lambda_n} \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r)\, r\, dr.$$

Bessel functions

Bessel equation of order $p \geq 0$:

$$r^2 y'' + r y' + (r^2 - p^2) y = 0, \qquad x > 0. \qquad \text{(BE)}$$

The general sol to (BE) of order p is:

$$y(r) = C_1 J_p(r) + C_2 Y_p(r)$$

If p is not an integer then a general sol is:

$$y(r) = C_1 J_p(r) + C_2 J_{-p}(r)$$

$J_p(r)$ = Bessel function of order p of the first kind

$Y_p(r)$ = " " " " p " " second kind

It is possible to derive series representations for Bessel functions:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}$$

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}, \quad p \text{ not an integer}$$

$n \in \mathbb{N}$ $Y_n(x)$ is constructed by a lim process: $Y_n(x) = \lim_{p \to n} Y_p(x)$

parenthesis?

Here $\Gamma(x)$ is the Gamma function.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$ (you don't need to know this)

The important properties of this function is that it behaves like the factorial and actually coincides with it at integer values:

$$\Gamma(x+1) = x\,\Gamma(x)$$ (analogous to $(n+1)! = (n+1)n!$)

$$\Gamma(n+1) = n!$$, n integer

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ (works even when factorial is not defined)

Back to Bessel functions: when $p = n \in \mathbb{N}$:

$$\boxed{J_{-n}(r) = (-1)^n J_n(r)}$$

Other properties of Bessel functions:

- If $p > 0$ $J_p(0) = 0$
- $\frac{d}{dr}\left[r^p J_p(r)\right] = r^p J_{p-1}(r)$ (~ recursion)
- $\frac{d}{dr}\left[r^{-p} J_p(r)\right] = -r^{-p} J_{p+1}(r)$

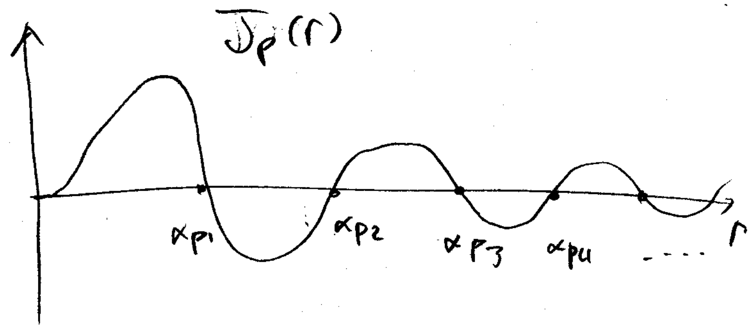
<u>Orthogonality of Bessel functions</u> (what makes them nice functions for doing expansions)

We can form a family of $\perp$ functions from J_p as follows:

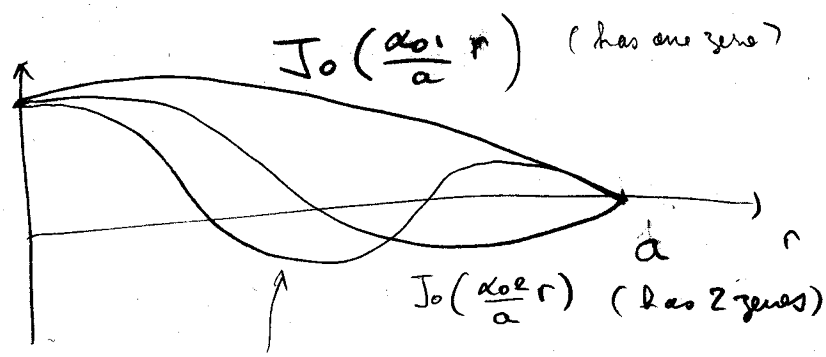
Consider the inner product: (weighted inner prod)

$$(u, v) = \int_0^a u(r)\, v(r)\, r\, dr$$

then: $$\left(J_p\left(\frac{\alpha_{pj}}{a} x\right), J_p\left(\frac{\alpha_{pk}}{a} x\right)\right) = \begin{cases} 0 & \text{if } k \neq j \\ \frac{a^2}{2} J_{p+1}^2(\alpha_{pj}) & \text{otherwise} \end{cases}$$



then the orthogonal Bessel functions are dilated s.t. they are zero at a: For $J_0(r)$ we get.



<u>Bessel series expansion</u>

If f is piece smooth on $[0, a]$

$$f(r) = \sum_{j=1}^{\infty} A_j J_p\left(\frac{\alpha_{pj}}{a} r\right)$$

where $$A_j = \frac{\left(f(r), J_p\left(\frac{\alpha_{pj}}{a} r\right)\right)}{\left(J_p\left(\frac{\alpha_{pj}}{a} r\right), J_p\left(\frac{\alpha_{pj}}{a} r\right)\right)}$$

Thus

$$\boxed{A_j = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a f(r)\, J_p\left(\frac{\alpha_{pj}}{a}\right) \mathbf{r}\, dr}$$

⚠ do not forget that this inner product is WEIGHTED.