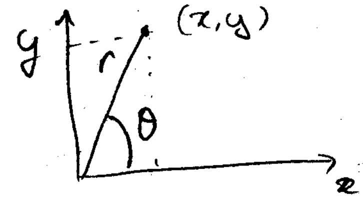
§4 PDE in polar and cylindrical coordinates



- **Polar coordinates**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \leftrightarrow \begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{x}{y} \end{cases}$$

$\theta = \tan^{-1}\frac{y}{x} + k\pi$,
where k depends on x and y.

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

(all terms have same units as r^{-2}).

is the representation of Laplacian in polar coordinate.
Deriving is long and tedious but it depends only on using chain rule and knowing derivatives such as:

$r = \sqrt{x^2+y^2}$ → $\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta}$, $\boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta}$

$\theta = \tan^{-1}\frac{y}{x}$ → $\boxed{\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} = -\frac{\sin\theta}{r}}$

$\boxed{\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{x}{r^2} = \frac{\cos\theta}{r}}$

$$\boxed{\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}(\cos\theta) = -\frac{\partial \theta}{\partial x}\sin\theta = +\frac{\sin^2\theta}{r}}$$

$$\boxed{\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}(\sin\theta) = -\frac{\partial \theta}{\partial y}\cos\theta = \frac{\cos^2\theta}{r}}$$

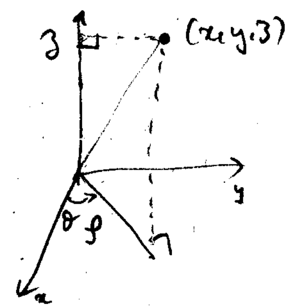
$$\boxed{\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = -\frac{\partial}{\partial x}(\sin\theta)\frac{1}{r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\sin\theta = 2\frac{\cos\theta\sin\theta}{r^2}}$$

$$\boxed{\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \frac{\partial}{\partial y}(\cos\theta)\frac{1}{r} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\cos\theta = -2\frac{\cos\theta\sin\theta}{r^2}}$$

(these are included to practice with changes of coordinates, see p195 for more details)

- Cylindrical coordinates

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$$



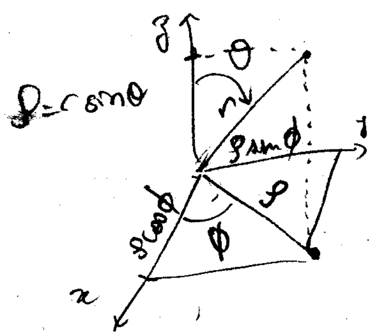
In cartesian coordinates: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

using polar form of Laplacian:

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Spherical coordinates

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases}$$



we have:
$r^2 = x^2 + y^2 + z^2$

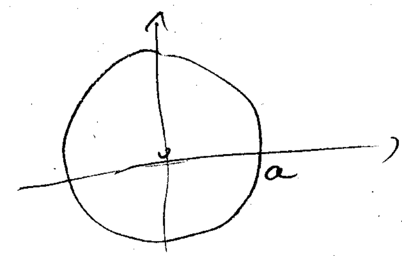
$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)}$$

Derivation involves using Laplac polar form on xy-plane and $z\rho$ plane, where ρ is projection of (x,y,z) onto (x,y) plane.

see p 137. We are not going to cover §5 so we won't use this too much in this class.

§ 4.2 Vibrations of a circular membrane (radially symm)

We study vibrations of a circular membrane (drum) clamped on edges i.e. $u(a,\theta,t)=0$, where $u(r,\theta,t)$ = displacement of membrane from equilibrium.



These are governed by the 2D wave equation:

$$u_{tt} = c^2 \Delta u$$

which in polar coordinates (better adapted to this problem than cartesian)

$$\boxed{u_{tt} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)}$$

We consider first the case where: (radially ~~symm~~)

initial shape $f(r,\theta) = f(r)$

initial velocity $g(r,\theta) = g(r)$

Because of the symmetries of the problem, $u(r,\theta) = u(r)$

$$\Rightarrow \frac{\partial u}{\partial \theta} = 0$$

Thus we are left with the simplified PDE:

$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) \\ u(a,t) = 0 \\ u(r,0) = f(r) \\ u_t(r,0) = g(r) \end{cases}$$

Use separation of variables: $u(r,t) = R(r)T(t)$

$$RT'' = c^2\left(R''T + \frac{1}{r}R'T\right)$$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{1}{R}\left(R''T + \frac{1}{r}R'\right) = -\lambda^2 = \text{const}$$

otherwise no time periodic sol!

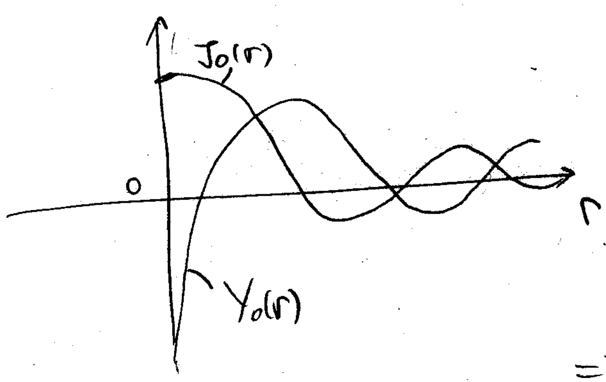
We get 2 equations to solve:

$$(1)\ \begin{cases} rR'' + R' + \lambda^2 rR = 0 \\ R(a) = 0 \end{cases} \qquad (2)\ \begin{cases} T'' + c^2\lambda^2 T \\ \dots \end{cases}$$

(1) Bessel equation of order 0 (parameter form)
2nd order linear eq $\Rightarrow$ we need 2 lin indep sol to describe all possible solutions.

- $J_0(\lambda r)$ Bessel function of order 0 of first kind
- $Y_0(\lambda r)$ ——————— 2nd ———

Sol to (1) is
$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$



$r \to 0 \Rightarrow Y_0(r) \to -\infty$
Since $|R(0)| < \infty$
we must have $c_2 = 0$

$\Rightarrow R(r) = c_1 J_0(\lambda r)$, take $c_1 = 1 \neq 0$
$R(a) = c_1 J_0(\lambda a) = 0$

λa must be a root of Bessel fun J_0.

Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ be zeros of J_0

then $\boxed{\lambda = \lambda_n = \dfrac{\alpha_n}{a}}$, $n = 1, 2 \dots$

and $\boxed{R_n(r) = J_0\left(\dfrac{\alpha_n}{a} r\right)}$, $n = 1 \dots$

(analogous to the ones in Sine Series expansion:
we solve $\sin(\lambda L) = 0 \Leftrightarrow \lambda L = n\pi$
$\lambda_n = \dfrac{n\pi}{L}$,

and analogy does not stop here!

Then we go back to (2) and write its solution:

$$T_n(t) = A_n \cos\left(\frac{c\,\alpha_n}{a} r\right) + B_n \sin\left(\frac{c\,\alpha_n}{a} r\right)$$

→ By construction:

$$\boxed{u_n(r,t) = R_n(t) T_n(t)} = \boxed{\left(A_n \cos\left(\frac{c\,\alpha_n}{a} r\right) + B_n \sin\left(\frac{c\,\alpha_n}{a} r\right)\right) J_0\left(\frac{\alpha_n}{a} r\right)}$$

$n = 1, 2, \dots$

solves DE and so does:

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos(c\lambda_n t) + B_n \sin(c\lambda_n t)\right) J_0(\lambda_n r)$$

What about B.C.?

$$f(r) = u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \quad (*) = \text{Bessel Series Expansion}$$

Looks a lot like Fourier series expansion, and it relies on $J_0(\lambda_n r)$ being orthogonal in the inner prod:

$$(u,v) = \int_0^a u(r)\, v(r)\, r\, dr \qquad \left(\text{compare to } (u,v) = \int_{-p}^{p} u(x)\, v(x)\, dx\right.$$

and:

$$\left(J_0(\lambda_n r), J_0(\lambda_m r)\right) = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{a^2 J_1^2(\alpha_n)}{2} & \text{if } n = m \end{cases}$$

thus: dotting by $J_0(\lambda_n r)$ on both sides of $(*)$

$$A_n = \frac{(f(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r)\, J_0(\lambda_n r)\, r\, dr$$

here $J_1(r)$ = Bessel fun of order 1 (available in matlab ...