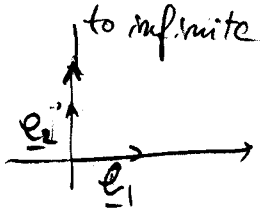
11

Note: this is just a generalization of what happens in the plane to infinite dimensions.



$\underline{e}_1 = (1,0)$, $\underline{e}_2 = (0,1)$ for a basis of $\mathbb{R}^2$.

Then any vector in $\mathbb{R}^2$ can be expanded in this basis:

$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$; here u_i = coeff of $\underline{u}$ in basis $\{\underline{e}_1, \underline{e}_2\}$

how to find the coeff?

$$(\underline{u}, \underline{e}_1) = (u_1 \underline{e}_1 + u_2 \underline{e}_2, \underline{e}_1)$$
$$= u_1 (e_1, e_1) + u_2 \underbrace{(e_2, e_1)}_{=0 \text{ (orth.)}}$$

$$\Rightarrow \quad u_i = \frac{(u, e_i)}{(e_i, e_i)}$$ "Fourier Coeff"

Same idea works with functions:

$$a_0 = \frac{(f, 1)}{(1,1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{(f, \cos nx)}{(\cos nx, \cos nx)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{(f, \sin nx)}{(\sin nx, \sin nx)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Proof: we do it only for a_m, $m \geq 1$: Dot f with corresp. basis function:

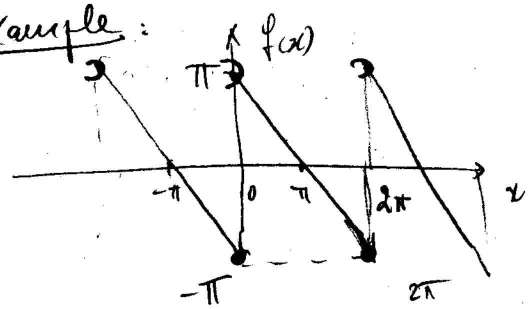
$$(f, \cos mx) = \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \cos mx \right)$$

$$= a_m (\cos mx, \cos mx) + \underbrace{\cancel{a_0 (1, \cos mx)}}_{0} + \sum_{\substack{n=1 \\ n \neq m}}^{\infty} a_n \cancel{(\cos nx, \cos mx)}$$
$$+ \sum_{n=1}^{\infty} b_n \underbrace{\cancel{(\sin nx, \cos mx)}}_{0}$$

$$= a_m (\cos mx, \cos mx)$$

Note: If f is 2π-periodic we can rewrite formulas by integrating say in $[0, 2\pi]$.

Example:



$f(x) = \pi - x \quad 0 \le x \le 2\pi$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$\overset{IBP}{=} \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n}\,dx \right]$$

$$= 0$$

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx}$$

$$\overset{IBP}{=} \frac{1}{\pi}\left[-(\pi - x)\frac{\cos nx}{n}\Big|_0^{2\pi} - \cancel{\int_0^{2\pi} \frac{\cos nx}{n}\,dx} \right]$$

will explain why "="

$$= \frac{1}{\pi}\left[\frac{\pi\cos 2\pi n}{n} + \frac{\pi\cos 0}{n} \right] = \boxed{\frac{2}{n}} \Rightarrow \boxed{f(x) \text{ "=" } \sum_{n=1}^{\infty} \frac{2}{n}\sin nx}$$

Show Falstad applet, explain how this decomposing a signal into "pure tones".

Show Gibbs phenomenon: overshooting at disc. points of partial Fourier series $\rightarrow$ $S_N(x) = a_0 + \sum_{i=1}^{N} a_n\cos + b_n\sin$

show Matlab code
Sawtooth.m

Weird thing about this approx:

$\sin n0 = 0 \quad$ So $\quad S_N(0) = 0$ whereas $f(0) \neq 0$.
$= \frac{f(0+) + f(0-)}{2}$.

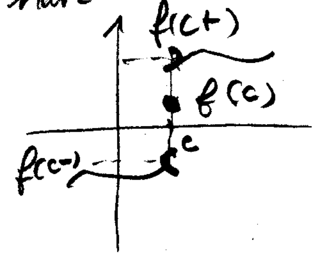
Fourier series repr: Let f be a 2π-per pcws smooth fn.

Then $\forall x$ we have:

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

So we get ptws convergence as long as we "hack" the function so that at every pt of discontinuity we have:

$$f(c) = \frac{f(c+)+f(c-)}{2}$$

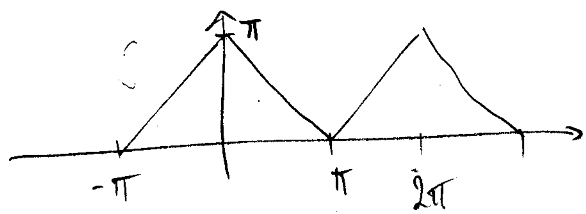


If we do this then our original expression:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{works.}$$

<u>Note</u>: If $f(x)$ is in addition continuous, we have no Gibbs phen

<u>Example</u>: Triangular wave



$$g(x) = \begin{cases} \pi - x & \text{if } 0 \le x \le \pi \\ \pi + x & \text{if } -\pi \le x \le 0 \end{cases}$$

<u>Fourier Coeff</u>: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\text{area of triangle } \frac{(2\pi)\times\pi}{2}$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)\cos nx}^{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$\stackrel{\text{IBP}}{=} \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left(\frac{-\cos nx}{n^2}\right)\Big|_0^{\pi} = \frac{2}{\pi n^2}(1 - \cos n\pi)$$

$$= \frac{2}{\pi n^2}(1 - (-1)^n) = \begin{cases} \frac{4}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\overbrace{f(x)\sin nx}^{\text{odd}}\,dx = 0$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi n^2}(1-(-1)^n)\cos nx = \sum_{k=0}^{\infty}\frac{4}{\pi(2k+1)^2}\cos((2k+1)x)$$

Show falstad applet. No Gibbs phenomenon.

Linear combinations of Fourier series

If $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

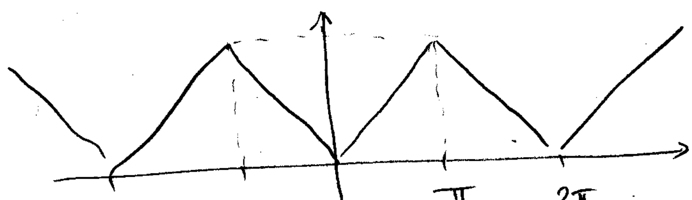
$$g(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx + \tilde{b}_n \sin nx$$

Then $(f+g)(x) = (a_0 + \tilde{a}_0) + \sum_{n=1}^{\infty} (a_n + \tilde{a}_n) \cos nx + (b_n + \tilde{b}_n) \sin nx$

See example 2.2.4 in textbook

Change of variables : Computing Fourier series of translated triangular wave

$h(x) = |x|$ ~~for~~ $\leq \pi$



using F.S. of $g(x)$

Note that: $h(x) = g(x+\pi)$

$$h(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} (1-(-1)^n) \cos n(x+\pi)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} (1-(-1)^n)(-1)^n \cos nx$$

$$\cos(n(x+\pi)) = \cos nx \underbrace{\cos n\pi}_{(-1)^n} - \cancel{\sin nx \sin n\pi}$$

Fourier coeff of $h(x)$ are:

$a_0 = \frac{\pi}{2}$; $a_n = \frac{2}{\pi n^2}((-1)^n - 1)$