Useful formulas

0.1. Complex numbers.

If $z = a + ib$, where $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$, then $\bar{z} = a - ib$ and $|z|^2 = \bar{z}z = a^2 + b^2$.
Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.
$\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$.

0.2. Fourier transforms.

$f(x)$	$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-i\omega x}$
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \widehat{f}(\omega) e^{i\omega x}$	$\widehat{f}(\omega)$
$\begin{cases} 1 & \text{if } \lvert x\rvert \le 1 \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$
$\frac{1}{a^2 + x^2}$, $a > 0$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\omega\rvert}}{a}$
$e^{-a\lvert x\rvert}$, $a > 0$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}$
$e^{-ax^2/2}$, $a > 0$	$\frac{1}{\sqrt{a}} e^{-\omega^2/(2a)}$
$f^{(n)}(x)$	$(i\omega)^n \widehat{f}(\omega)$
$x^n f(x)$	$i^n \frac{d^n}{d\omega^n} \widehat{f}(\omega)$
$(f * g)(x)$	$\widehat{f}(\omega)\widehat{g}(\omega)$
$\delta_0(x)$	$\frac{1}{\sqrt{2\pi}}$

0.3. Convolution.

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(x - y)\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - t) g(t)\, dt = (g * f)(x).$$

0.4. Fourier sine and cosine transforms.

$$\mathcal{F}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos\omega t\, dt \quad \text{and} \quad \mathcal{F}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\omega t\, dt.$$

If f is even $\mathcal{F}_c(f)(\omega) = \widehat{f}(\omega)$. If f is odd $\mathcal{F}_s(f)(\omega) = i\widehat{f}(\omega)$

0.5. Fourier series. For a $2p$−periodic piecewise smooth function f,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n x + b_n \sin\omega_n x,$$

where $\omega_n = n\pi/p$ and

$$a_0 = \frac{(f, 1)}{(1, 1)},\ a_n = \frac{(f, \cos\omega_n x)}{(\cos\omega_n x, \cos\omega_n x)},\ \text{and } b_n = \frac{(f, \sin\omega_n x)}{(\sin\omega_n x, \sin\omega_n x)}.$$

The inner product is $(u,v) = \int_{-p}^{p} u(x)v(x)dx$. The orthogonality relations are

$$(\cos\omega_n x, \cos\omega_m x) = \begin{cases} 2p & \text{if } n = m = 0 \\ p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases},$$

$$(\cos\omega_n x, \sin\omega_m x) = 0,$$

$$(\sin\omega_n x, \sin\omega_m x) = \begin{cases} p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases}.$$

0.6. **Integration formulas.**

$$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

$$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$

0.7. **Hyperbolic trigonometry.**

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$(\cosh x)' = \sinh x, \qquad (\sinh x)' = \cosh x$$

$$\cosh^2 x - \sinh^2 y = 1, \quad \sinh 0 = 0, \cosh 0 = 1$$

0.8. **Bessel functions.** The following identities are valid for $p \geq 0$ and $n = 0, 1, \ldots$.

$$\int J_1(r)dr = -J_0(r) + C \quad \text{and} \quad \int r^{p+1}J_p(r)dr = r^{p+1}J_{p+1}(r) + C$$

For $k \geq 0$, $a > 0$ and $\alpha > 0$, we have

$$\int_0^a (a^2 - r^2)r^{k+1}J_k(\frac{\alpha}{a}r)\,dr = 2\frac{a^{k+4}}{\alpha^2}J_{k+2}(\alpha).$$

0.9. **Orthogonality relations for Bessel functions.** Let $a > 0$ and $m \geq 0$ be fixed. Denote with α_{mn} the $n-$th positive zero of the Bessel function of the first kind of order m. With the inner product

$$(u,v) = \int_0^a u(r)v(r)r\,dr$$

we have for all j, k non-zero integers,

$$\Big(J_m(\frac{\alpha_{mj}}{a}r), J_m(\frac{\alpha_{mk}}{a}r)\Big) = \begin{cases} \frac{a^2}{2}J_{m+1}^2(\alpha_{mj}) & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

0.10. **Generalized functions (distributions).**

Dirac delta distribution δ_α: for smooth test functions ϕ: $\langle\delta_\alpha, \phi\rangle = \phi(0)$.

Heaviside distribution

$$\mathcal{U}_\alpha(x) = \begin{cases} 1 & \text{if } x \geq \alpha \\ 0 & \text{otherwise} \end{cases},$$

$$(\mathcal{U}_\alpha)' = \delta_\alpha$$