Math 3150-1, Midterm Exam 3
December 3 2008

Name: SOLUTIONS uNID: ____________

Each problem is independent.

Total points: 110/100.

Problem 1 (25 pts) Consider the Dirichlet problem on the unit disk,

$$\begin{cases} \Delta u = 0, & 0 < r < 1, \quad 0 < \theta < 2\pi \\ u(1,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases} \tag{1}$$

Recall that

- The Laplacian in polar coordinates is $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.
- A general form of the solution to the ODE $x^2y'' + xy' \underline{-} \rho^2 y = 0$ is

$$y(x) = c_1 x^\rho + c_2 x^{-\rho}, \quad \text{if } \rho \neq 0,$$
$$y(x) = c_1 \cancel{x} + c_2 \ln x, \quad \text{if } \rho = 0.$$

(a) Use separation of variables with $u(r,\theta) = R(r)\Theta(\theta)$ to show that the separated equations are of the form

$$r^2R'' + rR' - \lambda R = 0, \tag{2}$$
$$\Theta'' + \lambda\Theta = 0. \tag{3}$$

Plugging ansatz into (1) we get:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\xrightarrow{(R\Theta)^{-1}} \iff \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$$\iff \underbrace{r^2\frac{R''}{R} + r\frac{R'}{R}}_{\text{depends on } r \text{ only}} + \underbrace{\frac{\Theta''}{\Theta}}_{\text{depends on } \theta \text{ only}} = 0 \implies \begin{cases} r^2\frac{R''}{R} + r\frac{R'}{R} = \lambda \\ -\frac{\Theta''}{\Theta} = \lambda \end{cases}$$

$$\iff \boxed{\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}}$$

(b) Since Θ needs to be 2π-periodic, $\lambda = n^2$, $n = 0, 1, 2, \ldots$.
Solve equations (2) and (3).

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, 2 \ldots$$

$$R_n(r) = \begin{cases} c_1 r^n + c_2 r^{-n} & \text{if } n \neq 0 \\ c_1 + c_2 \ln r & \text{if } n = 0 \end{cases}$$

Since $\ln r$ an $r^{-n} \to \infty$ as $r \to 0$, we must have $c_2 = 0$.

(c) Show that the general form of the solution to the Dirichlet problem (1) is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Specify what are the coefficients a_n and b_n in terms of $f(\theta)$.

The products $u_n(r,\theta) = R_n(r)\,\Theta_n(\theta)$ solve (1) and so does their sum.

$$u(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\,\Theta_n(\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Since $u(1,\theta) = f(\theta)$ we have

$$a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta = f(\theta)$$

$\Rightarrow$ a_n & b_n are Fourier series coeff of $f(\theta)$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\,d\theta \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\,d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\,d\theta$$

(d) Solve the Dirichlet problem (1) with $f(\theta) = \sin(2\theta)$.

$f(\theta)$ is already given as it's Fourier series with $a_n = 0$, $n = 0, 1, \dots$
$b_2 = 1$, $b_n = 0$, $n \neq 2$

Thus: $\boxed{u(r,\theta) = r^2 \sin 2\theta}$

(e) [Extra credit] Write the solution to (d) in Cartesian coordinates.

$u(r,\theta) = r^2\, 2\cos\theta \sin\theta = \boxed{2xy.}$

Problem 2 (25 pts)
(a) Let $a \in \mathbb{R}$. Show that

$$\mathcal{F}(\exp[iax]f(x))(\omega) = \widehat{f}(\omega - a), \quad \text{and}$$
$$\mathcal{F}(\exp[-iax]f(x))(\omega) = \widehat{f}(\omega + a).$$

$$\boxed{\mathcal{F}(e^{iax} f(x))(\omega)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-i\omega x} e^{iax} f(x)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ix(\omega - a)} f(x)$$
$$\boxed{= \hat{f}(\omega - a)}$$

Letting $a = -a$ above we get second identity.

(b) Use (a) and Euler's formula to show that

$$\mathcal{F}(\cos(ax)f(x))(\omega) = \frac{\widehat{f}(\omega - a) + \widehat{f}(\omega + a)}{2}, \quad \text{and}$$
$$\mathcal{F}(\sin(ax)f(x))(\omega) = \frac{\widehat{f}(\omega - a) - \widehat{f}(\omega + a)}{2i}.$$

Since $\cos(ax) f(x) = \dfrac{e^{iax} + e^{-iax}}{2} f(x)$

we have $\boxed{\mathcal{F}(\cos(ax) f(x))(\omega)} = \frac{1}{2} \mathcal{F}(e^{iax} f(x))(\omega) + \frac{1}{2} \mathcal{F}(e^{-iax} f(x))(\omega)$

$$\boxed{= \frac{\hat{f}(\omega - a) + \hat{f}(\omega + a)}{2}}$$

Since $\sin(ax) f(x) = \dfrac{e^{iax} - e^{-iax}}{2i} f(x)$ we have

$$\mathcal{F}(\sin(ax) f(x))(\omega) = \frac{1}{2i} \mathcal{F}(e^{iax} f(x))(\omega) - \frac{1}{2i} \mathcal{F}(e^{-iax} f(x))(\omega)$$
$$= \frac{\hat{f}(\omega - a) - \hat{f}(\omega + a)}{2i}$$

(c) Use the identities (b) to find the Fourier transforms of

$$f(x) = \cos(x)\exp[-x^2] \quad \text{and}$$
$$g(x) = \frac{\sin(3x)}{1+x^2}.$$

we have $\mathcal{F}(e^{-x^2})(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$

$$\Rightarrow \boxed{\mathcal{F}(\cos x\, e^{-x^2})(\omega) = \frac{1}{2\sqrt{2}}\left[e^{-(\omega-1)^2/4} + e^{-(\omega+1)^2/4}\right]}$$

Since $\mathcal{F}\left(\frac{1}{1+x^2}\right)(\omega) = \sqrt{\frac{\pi}{2}}\, e^{-|\omega|}$

$$\Rightarrow \boxed{\mathcal{F}\left(\frac{\sin 3x}{1+x^2}\right)(\omega) = \frac{1}{2i}\sqrt{\frac{\pi}{2}}\left(e^{-|\omega-3|} - e^{-|\omega+3|}\right)}.$$

Problem 3 (10 pts)
(a) Compute the Fourier transform of $f(x) = x\exp[-x^2/2]$.
(b) What is the Fourier sine transform of $f(x)$ $(x \geq 0)$? (No calculations needed)

(a) Two easy ways:

$$f(x) = -\left(e^{-x^2/2}\right)'$$
$$\Rightarrow \hat{f}(\omega) = -i\omega\, e^{-\omega^2/2}$$

$$\mathcal{F}(x e^{-x^2/2})(\omega) = i\frac{d}{d\omega} e^{-\omega^2/2}$$
$$= -i\omega\, e^{-\omega^2/2}$$

(b) f is an odd function of x thus:

$$\boxed{\mathcal{F}_s(f)(\omega) = i\,\mathcal{F}(f)(\omega) = \omega\, e^{-\omega^2/2}}$$

Problem 4 (25 pts) Use the Fourier transform method to solve

$$\begin{cases} u_t(x,t) = t u_{xx}(x,t), & x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R}. \end{cases}$$

The solution should be given in terms of x and t.
Hint: Solutions to the ODE $y' + axy = 0$ have the form $y(x) = C\exp[-ax^2/2]$.

$$\mathcal{F} \quad \begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = -t\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

then $\hat{u}(\omega,t) = C(\omega)\, e^{-\omega^2 t^2/2}$

$\hat{u}(\omega,0) = C(\omega) = \hat{f}(\omega)$

$\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-\omega^2 t^2/2}$

$$\mathcal{F}^{-1} \quad \boxed{u(x,t) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\, \hat{f}(\omega)\, e^{-\omega^2 t^2/2}}$$

Problem 5 (25 pts) Consider the heat equation

$$\begin{cases} u_t = \frac{1}{4} u_{xx}, \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = f(x), \ x \in \mathbb{R} \end{cases} \tag{4}$$

(a) Use the Fourier transform method to show that the solution to the heat equation (4) is $u(x,t) = (g_t * f)(x)$, where

$$g_t(x) = \sqrt{\frac{2}{t}} \exp[-x^2/t].$$

$\mathcal{F}$

$$\begin{cases} \frac{d}{dt} \hat{u}(\omega, t) = -\frac{\omega^2}{4} \hat{u}(\omega, t) \\ \hat{u}(\omega, 0) = \hat{f}(\omega) \end{cases} \Rightarrow \hat{u}(\omega, t) = C(\omega) e^{-(\omega^2/4)t}$$

using I. C. $C(\omega) = \hat{f}(\omega)$

$$\hat{u}(\omega, t) = \hat{f}(\omega)\, e^{-\omega^2/(4t)}$$
$$= \hat{f}(\omega)\, \mathcal{F}\left(\sqrt{\frac{2}{t}} e^{-x^2/t}\right)$$

$\Rightarrow$ $\boxed{u(x,t) = f * g_t}$

since $\mathcal{F}(f*g) = \hat{f}\hat{g}$.

(b) Use (a) to show that the solution to heat equation (4) with

$$f(x) = \begin{cases} 20 & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

is

$$u(x,t) = 10\left[\operatorname{erf}\left(\frac{1-x}{\sqrt{t}}\right) + \operatorname{erf}\left(\frac{1+x}{\sqrt{t}}\right)\right],$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp[-u^2]\,du.$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\, g_t(x-z)\,dz$$

$$= \frac{20}{\sqrt{2\pi}}\int_{-1}^{1} g_t(x-z)\,dz$$

$$= \frac{20}{\sqrt{2\pi}}\sqrt{\frac{2}{t}}\int_{-1}^{1} e^{-(x-z)^2/t}\,dz$$

c.o.v. $u = (z-x)/\sqrt{t}$, $du = dz/\sqrt{t}$.

$$= \frac{20}{\sqrt{2\pi}}\sqrt{\frac{2}{t}}\int_{(-1-x)/\sqrt{t}}^{(1-x)/\sqrt{t}} \sqrt{t}\,du\, e^{-u^2}$$

$$= \frac{20}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\left(\frac{(1-x)}{\sqrt{t}}\right) + \operatorname{erf}\left(\frac{1+x}{\sqrt{t}}\right)\right]$$

$$= 10\left[\qquad \text{''} \qquad\right].$$