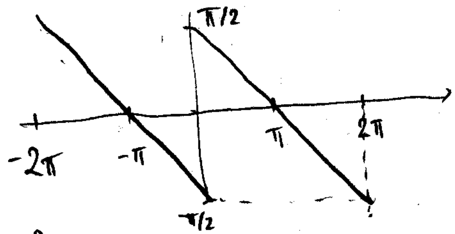
Problem 1

(a) The function $f(\theta)$ is defined for $\theta \in (0, 2\pi)$ but is a 2π-periodic function. It is odd so $\boxed{a_n = 0, \quad n = 0, 1, 2 \ldots}$.



$f(\theta) = \frac{\pi - \theta}{2}$, for $\theta \in (0, 2\pi)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$= \frac{\cancel{2}}{\pi} \int_0^{\pi} \frac{\pi - \theta}{\cancel{2}} \sin n\theta \, d\theta$$

$$= \int_0^{\pi} \sin n\theta \, d\theta - \frac{1}{\pi} \int_0^{\pi} \theta \sin n\theta \, d\theta$$

$$= -\frac{\cos n\theta}{n} \Big|_0^{\pi} - \frac{1}{\pi} \left[\frac{1}{n^2} \sin n\theta - \frac{\theta}{n} \cos n\theta \right]_0^{\pi}$$

$$= \frac{-(-1)^n + 1}{n} + \frac{1}{\pi} \left[\frac{\pi}{n} (-1)^n - 0 \right] = \frac{1}{n}$$

$$\Rightarrow \boxed{f(\theta) = \sum_{n=1}^{\infty} \frac{1}{n} \sin n\theta}$$

(b) The solution to the Dirichlet problem on the disk:

$$(*) \begin{cases} \Delta u = 0 & 0 < r < 1, \ 0 < \theta < 2\pi \\ u(1, \theta) = f(\theta), & 0 < \theta < 2\pi \end{cases}$$

is then:

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} \frac{r^n}{n} \sin n\theta}$$

This is derived by doing separation of variables in $(*)$ and $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$

$$u = R(r)\Theta(\theta) \Rightarrow \quad R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$

$$\Leftrightarrow \quad r^2 \frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$r^2 R'' + rR' - \lambda R = 0$ is Euler's equation and has solutions:

$$R(r) = c_1 \left(\frac{r}{a}\right)^n + \underbrace{c_2 \left(\frac{r}{a}\right)^{-n}}_{\text{non physical so } c_2 = 0.}$$

Problem 2

(a)

$$\boxed{\mathcal{F}(f(ax))(\omega)} = \int_{-\infty}^{\infty} dx\, e^{-i\omega x} f(ax) \underset{\substack{\uparrow \\ \text{cov} \\ u = ax \\ du = a\,dx \\ (a\neq 0)}}{=} \underbrace{\operatorname{sgn}(a)\int_{-\infty}^{\infty}}_{\substack{\text{a negative will} \\ \text{reverse integration} \\ \text{bounds.}}} \frac{du}{a}\, e^{-i\omega u/a} f(u)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} du\, e^{-i(\omega/a)u} f(u) = \boxed{\frac{1}{|a|}\mathcal{F}(f)(\omega/a)}$$

(b) We have $f(x) = e^{-|x|} \Rightarrow \hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$

Thus $g(x) = e^{-5|x|} = f(5x)$

$$\boxed{\hat{g}(\omega) \underset{\substack{\uparrow \\ \text{by (a)}}}{=} \frac{1}{5}\hat{f}(\omega/5) = \frac{1}{5}\sqrt{\frac{2}{\pi}}\frac{1}{1+(\omega/5)^2} = \sqrt{\frac{2}{\pi}}\frac{5}{5^2+\omega^2}}$$

(c) We have $f(x) = e^{-x^2/2}$ and $g(x) = e^{-ax^2/2} = f(\sqrt{a}x)$ $(a>0)$

$$\boxed{\hat{g}(\omega) = \frac{1}{\sqrt{a}}\hat{f}(\omega/\sqrt{a}) = \frac{1}{\sqrt{a}}e^{-(\omega/\sqrt{a})^2/2} = \frac{1}{\sqrt{a}}e^{-\omega^2/2a}}$$

Problem 3

$$f(x) = \begin{cases} x & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases} = x\,g(x)$$

where $g(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i\omega x} g(x) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} dx\, e^{-i\omega x} = \frac{1}{\sqrt{2\pi}}\frac{1}{-i\omega} e^{-i\omega x}\Big|_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\frac{i}{\omega}\left(e^{-i\omega} - e^{i\omega}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{i}{\omega}(-2i\sin\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$ (or faster: using table)

$$\boxed{\mathcal{F}(f)(\omega) = \mathcal{F}(xg(x))(\omega) \underset{\substack{\uparrow \\ \text{see table}}}{=} i\frac{d}{d\omega}\hat{g}(\omega) = i\frac{d}{d\omega}\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega} = i\sqrt{\frac{2}{\pi}}\frac{\omega\cos\omega - \sin\omega}{\omega^2}}$$

Problem 4

$$\begin{cases} u_{tt} = u_{xxxx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

$\Big\downarrow \mathcal{F}$

$$\begin{cases} \frac{d^2}{dt^2} \hat{u}(\omega,t) = (i\omega)^4 \hat{u}(\omega,t) = \omega^4 \hat{u}(\omega,t) & (1) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \\ \frac{d}{dt}\hat{u}(\omega,0) = 0 \end{cases}$$

Solving ODE (1):

$$\hat{u}(\omega,t) = A(\omega)\cosh(\omega^2 t) + B(\omega)\sinh(\omega^2 t)$$

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\frac{d}{dt}\hat{u}(\omega,t) = \omega^2 \hat{f}(\omega)\sinh(\omega^2 t) + \omega^2 B(\omega)\cosh(\omega^2 t)$$

$$\frac{d}{dt}\hat{u}(\omega,0) = \omega^2 B(\omega) = 0 \Rightarrow B(\omega) = 0$$

$$\Rightarrow \boxed{\hat{u}(\omega,t) = \hat{f}(\omega)\cosh(\omega^2 t)}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\, \hat{f}(\omega)\cosh(\omega^2 t)$$

using the particular $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{+i\omega x}\cosh(\omega^2 t)\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$\boxed{= \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, e^{+i\omega x}\cosh(\omega^2 t)\,\frac{\sin\omega}{\omega}}$$

Problem 5

$$\begin{cases} u_t = c^2 u_{xx} + k u_x \\ u(x,0) = f(x) \end{cases}, \quad x \in \mathbb{R},\ t>0,\ k>0.$$

(Heat eq with convection)

$\downarrow \mathcal{F}$

$$\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = c^2 (i\omega)^2 \hat{u}(\omega,t) + k(i\omega)\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = (ik\omega - c^2\omega^2)\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\underset{\text{solving ODE}}{\Rightarrow} \hat{u}(\omega,t) = A(\omega) e^{(ik\omega - c^2\omega^2)t}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega) = A(\omega)$$

$$\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega) e^{(ik\omega - c^2\omega^2)t}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{(-c^2\omega^2 + ik\omega)t} e^{ix\omega}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega(x+kt)}\, d\omega$$

Problem 6 The Poisson kernel is: $P_y(x) = \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$

(a) $$\hat{P}_y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, P_y(x) e^{-i\omega x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2} e^{-i\omega x} dx$$

$$= \sqrt{\frac{2}{\pi}}\, y\, \mathcal{F}\left(\frac{1}{x^2+y^2}\right)(\omega) = \cancel{\sqrt{\frac{2}{\pi}}}\, \cancel{y}\, \cancel{\sqrt{\frac{\pi}{2}}}\, \frac{e^{-y|\omega|}}{\cancel{y}} = e^{-y|\omega|}$$

(b)

$$\mathcal{F}(P_y * P_z)(\omega) = \hat{P}_y(\omega)\hat{P}_z(\omega) = e^{-y|\omega|} e^{-z|\omega|} = e^{-(y+z)|\omega|} = \mathcal{F}(P_{y+z})(\omega)$$

$$\Rightarrow \boxed{P_y * P_z = P_{y+z}}$$