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Differential Equations and Linear Algebra 2250

Sample Midterm Exam 3

2014

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 5)** Complete all.

(1a) [50%] The differential equation $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 12x^2 + 6x$ has a particular solution $y_p(x)$ of the form $y = d_1x^2 + d_2x^3 + d_3x^4$. Find $y_p(x)$ by the method of undetermined coefficients (yes, find d_1, d_2, d_3).

Answer:

Solution y_h is a linear combination of the atoms $1, x, \cos(x), \sin(x)$. A particular solution is $y_p = x^4 + x^3 - 12x^2$.

The atoms for $y^(4) + y'' = 0$ are found from $r^4 + r^2 = 0$ with roots $r = 0, 0, i, -i$. The atoms in $f(x) = 12x^2 + 6x$ are $1, x, x^2$. Because $1, x$ are solutions of the homogeneous equation, then the list $1, x, x^2$ from $f(x)$ is multiplied by x^2 to obtain the corrected list x^2, x^3, x^4. Then $y_p = d_1x^2 + d_2x^3 + d_3x^4$.

Substitute y_p into the equation $y^(4) + y'' = 12x^2 + 6x$ to get $24d_3 + 2d_1 + 6d_2x + 12d_3x^2 = 12x^2 + 6x$. Matching coefficients of atoms gives $24d_3 + 2d_1 = 0, 6d_2 = 6, 12d_3 = 12$. Then $d_3 = 1, d_2 = 1, d_1 = -12$. Finally, $y_p = (-12)x^2 + (1)x^3 + (1)x^4$.

(1b) [20%] Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped. **To save time, do not solve for $x(t)$!**

Answer:

Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4)$, therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor $5r^2 + 2r + 4$ to obtain roots $(-1 \pm \sqrt{19}i)/5$ and then classify as **under-damped**.

(1c) [30%] Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

Answer:

The answer is the undetermined coefficients solution $x_p(t) = A\cos(5t) + B\sin(5t)$, because the homogeneous solution $x_h(t)$ has limit zero at $t = \infty$. Substitute the trial solution into the differential equation. Then $-8A\cos(5t) - 8B\sin(5t) - 10A\sin(5t) + 10B\cos(5t) = 82\sin(5t)$. Matching coefficients of sine and cosine gives the equations $-8A + 10B = 0$, $-10A - 8B = 82$. Solving, $A = -5, B = -4$. Then $x_p(t) = -5\cos(5t) - 4\sin(5t)$ is the unique periodic steady-state solution.

Use this page to start your solution. Attach extra pages as needed.

Name. ________________________________

2. (Chapter 5) Complete all.

(2a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots 0, 0, -1, -1, $2i$, $-2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 5e^{-x} + 4x^2 + x\cos 2x + \sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The Euler solution atoms for roots of the characteristic equation are $1, x, e^{-x}, xe^{-x}, \cos 2x, \sin 2x,$.
The atom list for $f(x)$ is e^{-x}, $1, x, x^2$, $\cos 2x$, $x\cos 2x$, $\sin 2x$, $x\sin 2x$. This list of 8 atoms is broken into 4 groups, each group having exactly one base atom: (1) $1, x, x^2$, (2) e^{-x}, (3) $\cos 2x$, $x\cos 2x$, (4) $\sin 2x$, $x\sin 2x$. Each group contains a solution of the homogeneous equation. The modification rule is applied to groups 1 through 4. The trial solution is a linear combination of the replacement 8 atoms in the new list (1*) x^2, x^3, x^4, (2*) x^2e^{-x}, (3*) $x\cos 2x$, $x^2\cos 2x$ (4*) $x\sin 2x$, $x^2\sin 2x$.

(2b) [40%] Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Answer:

The characteristic polynomial is the expansion $(r-1.2)^4((r+1)^2+1)^3$. Because x^3e^{ax} is an Euler solution atom for the differential equation if and only if $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$ are Euler solution atoms, then the characteristic equation must have roots $1.2, 1.2, 1.2, 1.2$, listing according to multiplicity. Similarly, $x^2e^{-x}\sin(x)$ is an Euler solution atom for the differential equation if and only if $-1 \pm i, -1 \pm i, -1 \pm i$ are roots of the characteristic equation. Total of 10 roots with product of the factors $(r-1)^4((r+1)^2+1)^3$ equal to the 10th degree characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.

Name. ______________________________

3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [40%] Display the details of Laplace's method to solve the system for $x(t)$. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + 3y, \\ y' &= -2y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-1)\mathcal{L}(x) + (-3)\mathcal{L}(y) = 1, \quad (0)\mathcal{L}(x) + (s+2)\mathcal{L}(y) = 2.$$

Any other method of arriving at the system of linear algebraic equations is acceptable, it is not an error to do it another way. Elimination or Cramer's rule or matrix inversion applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{(s+2)+6}{(s-1)(s+2)} = \dfrac{-2}{s+2} + \dfrac{3}{s-1}$. Then the backward table implies $x(t) = -2e^{-2t} + 3e^t$.

(3b) [30%] Find $f(t)$ by partial fractions, the shifting theorem and the backward table, given

$$\mathcal{L}(f(t)) = \frac{2s^3 + 3s^2 - 6s + 3}{s^3(s-1)^2}.$$

Answer:

The numerator has degree 3, less than the denominator degree 5. Partial fraction theory gives an expansion $\frac{2s^3+3s^2-6s+3}{s^3(s-1)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{s-1} + \frac{e}{(s-1)^2}$. We find the system of equations and solve it for $a = b = 0$, $c = 3$, $d = 0$, $e = 2$. Then $\mathcal{L}(f(t)) = \frac{3}{s^3} + \frac{2}{(s-1)^2} = \mathcal{L}(3t^2/2 + 2te^t)$ implies $f(t) = 3t^2/2 + 2te^t$.

(3c) [30%] Solve for $f(t)$, given

$$\mathcal{L}(e^{2t}f(t)) + 2\frac{d^2}{ds^2}\mathcal{L}(tf(t)) = \frac{s+3}{(s+1)^3}.$$

Answer:

Use the s-differentiation theorem, partial fractions and the backward Laplace table plus the shift theorem to get $\mathcal{L}(e^{2t}f(t)) + 2\mathcal{L}((-t)^2(t)f(t)) = \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} = \mathcal{L}(te^{-t} + t^2e^{-t})$. Lerch's theorem implies $(e^{2t} + 2t^3)f(t) = te^{-t} + (t^2)e^{-t}$. Then $f(t) = (t + t^2)e^{-t}/(e^{2t} + 2t^3)$.

Use this page to start your solution. Attach extra pages as needed.

Name. ______________________________

4. **(Chapter 10)** Complete all parts.

(4a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{-t}\cos(4t)$	
$(t+2)^2$	
t^2e^{-2t}	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{3}{s^2+9}$	$\sin 3t$
$\dfrac{s-1}{s^2-2s+5}$	
$\dfrac{2}{(2s-1)^2}$	
$\dfrac{s}{(s-1)^3}$	

Answer:

Forward: $\dfrac{s+1}{(s+1)^2+16}$, $\quad \dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}$, $\quad \frac{2}{(s+2)^3}$.

Backward: $e^t\cos(2t)$, $\quad (1/2)te^{t/2}$, $\quad te^t+(t^2/2)e^t$.

(4b) [20%]
Find $\mathcal{L}(f(t))$ from the Second Shifting theorem, given $f(t)=\sin(2t)\mathbf{u}(t-2)$, where $\mathbf{u}$ is the unit step function defined by $\mathbf{u}(t)=1$ for $t\ge 0$, $\mathbf{u}(t)=0$ for $t<0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t\to t+a}\right).$$

Write $g(t)=\sin(2t)$.

Then $\mathcal{L}(f(t)) = \mathcal{L}(g(t)u(t-2)) = e^{-2s}\mathcal{L}(g(t)|_{t->t+2}) = e^{-2s}\mathcal{L}(\sin(2t+4)) = e^{-2s}\mathcal{L}(\sin(2t)\cos(4)+\sin(4)\cos(2t))$. Then Lerch's law implies $\mathcal{L}(f(t)) = e^{-2s}\left(\frac{2\cos 4}{s^2+4}+\frac{s\sin(4)}{s^2+4}\right)$.

(4c) [20%] Find $f(t)$ from the Second Shifting Theorem, given $\mathcal{L}(f(t)) = \dfrac{s\,e^{-\pi s}}{s^2+2s+17}$.

Answer:

Use the second shifting theorem in backward form

$$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)\mathbf{u}(t-a)).$$

Then $\mathcal{L}(f(t)) = \dfrac{s\,e^{-\pi s}}{(s^2+2s+1)+16} = e^{-\pi s}\mathcal{L}(e^{-t}\cos(4t)) = \mathcal{L}(e^{-t}\cos(t)u(t)\Big|_{t\to(t-\pi)}) = \mathcal{L}(e^{-t+\pi}\cos(t-\pi)u(t-\pi))$. Then Lerch's theorem implies $f(t) = e^{-t+\pi}\left(\cos(t)\cos(\pi)+\sin(t)\sin(\pi)\right)u(t-\pi)$, which reduces to $f(t) = -e^{-t+\pi}\cos(t)u(t-\pi)$.

Use this page to start your solution. Attach extra pages as needed.

Name. ______________________________

5. **(Chapter 6)** Complete all parts.

(5a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 4 & 1 & 12 \\ -4 & 1 & -3 & 15 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & -2 & 7 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:

$1, 5, 1 \pm 4i$

(5b) [30%] Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, which has eigenvalues $1, 2, 2$, find all eigenvectors for eigenvalue 2.

Answer:

One frame sequence is required for $\lambda = 2$. The sequence starts with $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer $x_1 = -t_1 + t_2$, $x_2 = t_1$, $x_3 = t_2$. Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(5c) [20%] Suppose a 3×3 matrix A has eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:

Define $P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $AP = PD$.

(5d) [20%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the matrix C.

Answer:

The eigenvalues come from the exponents in the exponentials, 2 and 2. The eigenpairs are $\left(2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Solve $CP = PD$ to find $C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The usual eigenpairs for C are the columns of the identity. But the eigenvalues are equal, therefore any linear combination of the two eigenvectors is also an eigenvector. This justifies the correctness of the strange eigenpairs given in the problem.

Use this page to start your solution. Attach extra pages as needed.