Partial Differential Equations 3150
Final Exam
Fall 2009

Instructions: This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

Keep this page for your records, or discard it.

1. (Fourier Series)
Let $f(x) = -1$ on the interval $0 < x < \pi$, $f(x) = 1$ on $-\pi < x < 0$. Let $f(x$ be defined at $x = 0, \pi, -\pi$ to make its 2π-periodic extension $g(x)$ satisfy $g(-x) = g(x)$.

(a) [25%] Display the formulas for the Fourier coefficients of f, but do not calculate the integrals.

(b) [25%] Compute the first two nonzero Fourier coefficients.

(c) [25%] Let $g(x)$ equal the Fourier series of f. Calculate the value of $g(x)$ for every x in $-\infty < x < \infty$.

(d) [25%] Draw a figure for this example, to illustrate Gibb's phenomenon. Include detail about the Fourier series partial sum $S_N(x)$, where N is the first index that includes the terms found in part (b) above.

Use this page to start your solution. Attach extra pages as needed, then staple.

2. (CH3. Finite String: Fourier Series Solution)
Consider the problem

$$\begin{cases} u_{tt}(x,t) &=& c^2 u_{xx}(x,t), & 0 < x < 2\pi, & t > 0, \\ u(0,t) &=& 0, & & t > 0, \\ u(2\pi,t) &=& 0, & & t > 0, \\ u(x,0) &=& f(x), & 0 < x < 2\pi, & \\ u_t(x,0) &=& 0, & 0 < x < 2\pi & \end{cases}$$

(a) [50%] Do the analysis of product solutions $u = X(x)T(t)$.

(b) [25%] Display the series solution $u(x,t)$. Include formulas for the coefficients.

(c) [25%] Find the solution $u(x,t)$ when $c = 1/\pi$ and $f(x) = 7\sin(4x) - 8\sin(9x)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

3. (CH4. Steady-State Heat Conduction on a Disk)
Consider the problem

$$\begin{cases} u_{rr}(r,\theta) + \dfrac{1}{r}u_r(r,\theta) + \dfrac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < 2, \quad 0 < \theta < 2\pi, \\ u(2,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

(a) [75%] A product solution has the form $R(r)\Theta(\theta)$. Do the analysis to find formulas for $R(r)$ and $\Theta(\theta)$. Then display the series solution $u(x,t)$.

(b) [25%] Find the coefficients in the series solution when $f(\theta) = \theta$ on $0 < \theta < 2\pi$, $f(0) = f(2\pi) = 0$, $f(\theta + 2\pi) = f(\theta)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

4. (CH4. Poisson Problem)
Solve for $u(x,y)$ in the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} &= \sin(\pi x)\sin(\pi y), & 0 < x < 1, \quad 0 < y < 1, \\ u(x,0) &= \sin(\pi x) + 3\sin(4\pi x), & 0 < x < 1, \\ u(x,y) &= 0 & \text{on the other 3 boundary edges.} \end{cases}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

5. (CH4. Heat Equation)
Let $f(x) = x$ for $0 \le x < 1$ and $f(x) = 0$ elsewhere. Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t), & 0 < x < 1, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(1,t) &= 0, & & t > 0, \\ u(x,0) &= 1 & 0 < x < 1. & \end{cases}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

6. (CH7. Fourier Transform: Infinite Rod)
Let $f(x) = 1$ for $0 < x < 2$ and $f(x) = 0$ elsewhere on $-\infty < x < \infty$. Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \quad\quad\quad -\infty < x < \infty. \end{cases}$$

Use this page to start your solution. Attach extra pages as needed, then staple.