Midterm Two, Math 3150-3, Oct 18, 2007

Name: ______________________ **U. ID:** ____________

Instructions: This is a closed book but open written notes exam. Calculators are not allowed. You need to show all the details of your work to receive full credits.

Problem	1	2	3	4	& total
worth of points	25	25	25	25	100
your points					

1. Solve the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0.$$

$$u(0,t) = 0, \ u(1,t) = 0.$$

$$u(x,0) = \sin(3\pi x), \quad \frac{\partial u}{\partial t}(x,0) = \sin(\pi x) - 3\sin(8\pi x).$$

$$u = \sum_n (b_n \cos\lambda_n t + b_n^* \sin\lambda_n t) \sin n\pi x.$$

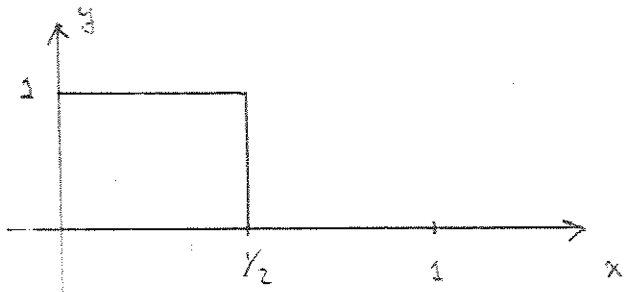
$$c = 1 \Rightarrow \lambda_n = n\pi$$

$$u = \cos 3\pi t \sin 3\pi x + \frac{1}{\pi}\sin\pi t \sin\pi x - \frac{3}{8\pi}\sin 8\pi t \sin 8\pi x$$

2. Use d'Alembert's formula to sketch the solution to the vibrating string problem at time $t = \frac{1}{2}$, subject to the following conditions:

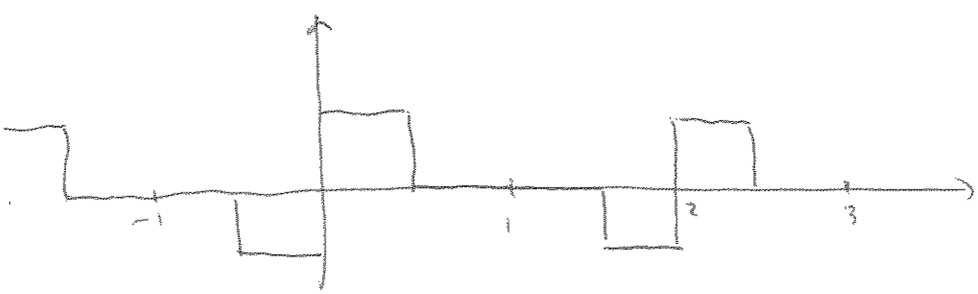
$$u(0,t) = u(1,t) = 0, \; g(x) = 0, \; c = 1,$$

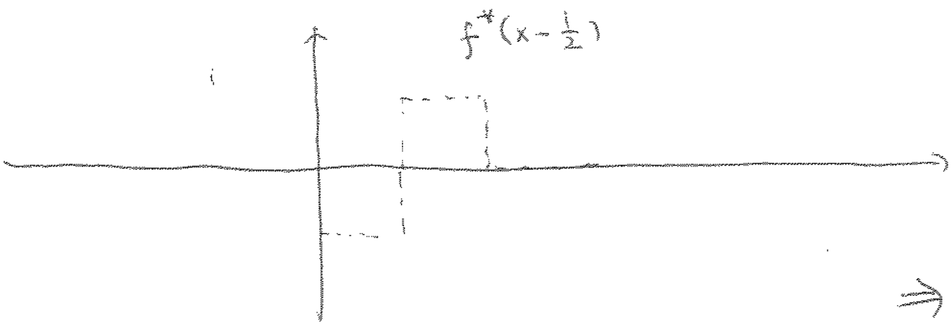
and the initial displacement $f(x)$ plotted below.



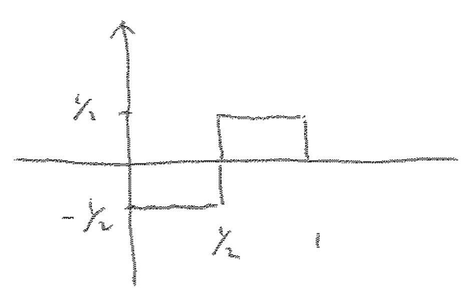
$$u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$$

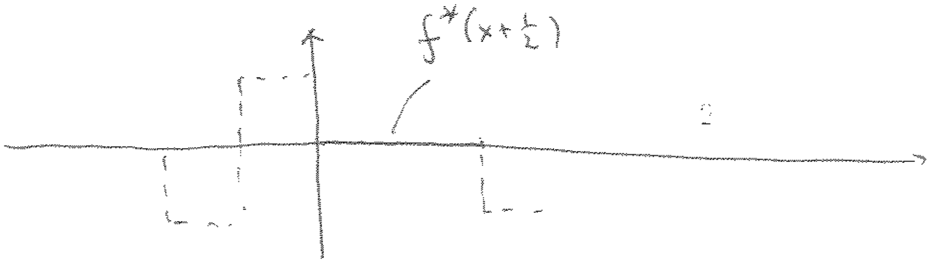
$$u(x, \tfrac{1}{2}) = \frac{1}{2}\left[f^*(x+\tfrac{1}{2}) + f^*(x-\tfrac{1}{2}) \right]$$





$\Rightarrow$





3. Solve the following heat equation problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0.$$

$$u(0,t) = 0, \quad u(1,t) = 1,$$

$$u(x,0) = x + \sin \pi x, \quad 0 < x < 1.$$

$u = u_1 + u_2$

$u_1 = x$

u_2 solves

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin \pi x$$

$$u_2 = e^{-\pi^2 t} \sin \pi x$$

$$\Rightarrow \quad u = x + e^{-\pi^2 t} \sin \pi x$$

4. Consider the heat conduction in a bar where the left end temperature is maintained at 0, and the right end is perfectly insulated. We assume $L = 1$ and $c = 1$.

(a) Derive the boundary conditions for the temperature at these two ends;

(b) Following the separation of variables approach, derive the ODEs for X and T;

(c) We will focus on the problem for $X(x)$. What are the boundary conditions for X at $x = 0$ and $x = 1$? Show that solutions of the form $X(x) = \sin \mu x$ satisfy the ODE for X and one of the boundary conditions. Can you choose certain values of μ so that the other boundary condition is also satisfied?

(a) $u(0, t) = 0$

$\frac{\partial u}{\partial x}(1, t) = 0$

(b) $X'' + \mu^2 X = 0$ $\quad X(0) = 0, \quad X'(1) = 0$

$T' + \mu^2 T = 0$

(c) $X = \sin \mu x. \quad X' = \mu \cos \mu x, \quad X'' = -\mu^2 \sin \mu x$

$X(0) = 0$

$X'(1) = \mu \cos \mu = 0.$

$\mu = \frac{n\pi}{2}, \quad n = \cancel{1,2,3}, 1, 3, 5, \ldots$