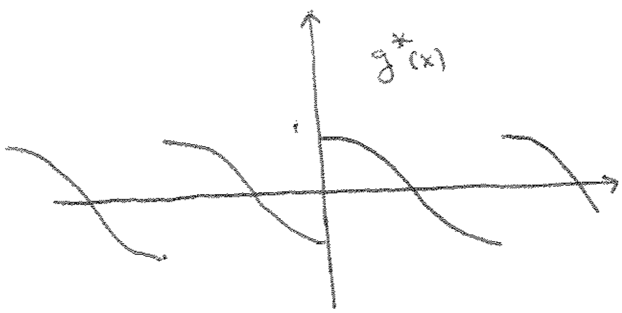
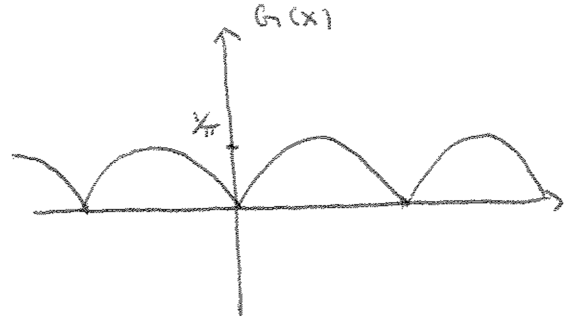
3.3.7 $$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx$$
$$= 2\left[\int_0^{1/4} 4x\sin n\pi x\,dx + \int_{1/4}^{3/4}\sin n\pi x\,dx + \int_{3/4}^{1} 4(1-x)\sin n\pi x\,dx\right]$$

$$b_n^* = \frac{2}{4n\pi}\int_0^1 1\cdot\sin\frac{n\pi x}{1}\,dx = \frac{1}{2n\pi}\int_0^1 \sin n\pi x\,dx$$
$$= \frac{1}{2(n\pi)^2}\left(1-\cos n\pi\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{1}{(n\pi)^2} & n \text{ odd} \end{cases}$$

3.4.6 $f^*(x) = 0$





3.5.13 $$u_1(x) = \frac{50-100}{\pi}x + 100$$

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t}\sin nx, \qquad \lambda_n = n$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left\{f(x) - \left[-\frac{50}{\pi}x + 100\right]\right\}\sin nx\,dx$$
$$= \frac{2}{\pi}\left[\int_0^{\pi/2} 33x\sin nx\,dx + \int_{\pi/2}^{\pi} 33(\pi - x)\sin nx\,dx\right]$$
$$- \frac{2}{\pi}\int_0^{\pi}\left(100 - \frac{50}{\pi}x\right)\sin nx\,dx$$
$$= c_n + d_n$$

$$c_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{132}{\pi}\dfrac{(-1)^k}{n^2} & n \text{ odd}, \quad n = 2k+1 \end{cases}$$

$$d_n = \frac{100}{\pi}\,\frac{2-(-1)^n}{n}$$

3.6.3 $$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos nx. \qquad \lambda_n = n$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = 25\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} 100x\cos nx\,dx$$

$$+ \frac{2}{\pi}\int_{\pi/2}^{\pi} 100(\pi - x)\cos nx\,dx$$

~~n even~~

$$= \frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)\cdot\frac{200}{\pi}$$

First, we note when n is odd, $a_n = 0$. Next we assume n is even, $\cos n\pi = 1$, so $a_n = \frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - 2\right)$

To be more specific, we discuss separately when $n/2$ is even or odd:

$$n = 2(2m): \quad \frac{\pi}{200} a_n = \frac{1}{16m^2}\left(2\cos 2m\pi - 2\right) = 0$$

$$n = 2(2m+1): \quad \frac{\pi}{200} a_n = \frac{1}{4(2m+1)^2}\left(2\cos(2m+1)\pi - 2\right)$$

$$= \frac{-4}{4(2m+1)^2} = -\frac{1}{(2m+1)^2}$$

Finally we have solution

$$u(x,t) = 25\pi - \frac{200}{\pi}\sum_{m=0}^{\infty} \frac{e^{-4(2m+1)^2 t}\cos 2(2m+1)x}{(2m+1)^2}$$

3.6.12 $$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-c^2\mu_n^2 t}\sin\mu_n x, \quad c = 1$$

$$c_n = \frac{1}{\int_0^1 \sin^2\mu_n x\,dx}\int_0^1 \sin\pi x\,\sin\mu_n x\,dx$$

You can use a computer to find the numerical values, but sketch like above is good enough in a test for this particular problem.