Section 4.4

4. $$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} (\pi-\theta)\, d\theta = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} (\pi-\theta)\cos n\theta\, d\theta = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi-\theta)\sin n\theta\, d\theta = \frac{1}{n}$$

$$\Rightarrow \boxed{u = \frac{\pi}{4} + \sum_{n=1}^{\infty} r^n \left[\frac{1-(-1)^n}{n^2\pi}\cos n\theta + \frac{1}{n}\sin n\theta \right]}$$

14

$$a = \pi/2$$

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin 2n\theta \qquad 0<r<1, \quad 0 \le \theta \le \pi/2$$

$$2n \cdot b_n = \frac{2}{\pi/2}\int_0^{\pi/2} \theta \sin 2n\theta\, d\theta = \frac{1}{n}(-1)^{n+1}$$

$$\Rightarrow b_n = \frac{(-1)^{n+1}}{2n^2}$$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^2} r^{2n} \sin 2n\theta}$$

Section 7.1

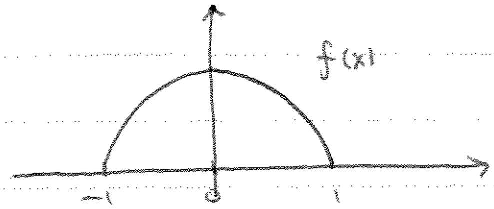
12. $$A(\omega) = \frac{1}{\pi}\int_0^{\infty} e^{-t}\cos \omega t\, dt = \frac{1}{\pi(1+\omega^2)}$$

$$B(\omega) = \frac{1}{\pi}\int_0^{\infty} e^{-t}\sin \omega t\, dt = \frac{\omega}{\pi(1+\omega^2)}$$

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty} \left(\frac{\cos \omega x}{1+\omega^2} + \frac{\omega \sin \omega x}{1+\omega^2} \right) d\omega}$$

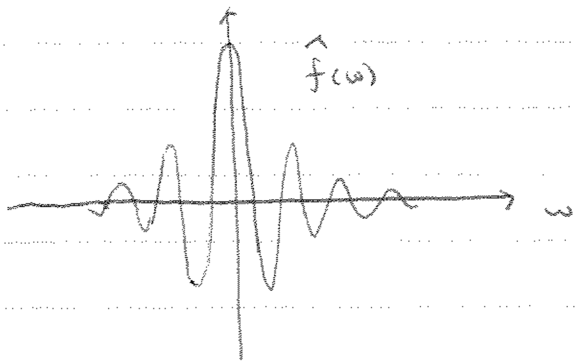
Section 7.2

2



$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\sin\omega}{\omega} + \frac{4\cos\omega}{\omega^2} - \frac{4\sin\omega}{\omega^3} \right]$$



19 $$\mathcal{F}(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{-i\omega x}\, dx \overset{x' = x-a}{=\!=\!=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-i\omega(x'+a)}\, dx'$$

$$= e^{-i\omega a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-i\omega x'}\, dx' = e^{-i\omega a}\, \mathcal{F}(f(x))$$

22 Write $f(x) = \sin 2x \cdot g(x)$, $g(x) = \frac{1}{e^{|x|}} = e^{-|x|}$

According to A66, $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2} = \hat{g}(\omega)$

Using the result in Exercise 20

$$\mathcal{F}(\sin 2x \cdot g(x)) = \frac{\hat{g}(\omega-2) - \hat{g}(\omega+2)}{2i}$$

$$= \frac{1}{i\sqrt{2\pi}} \left[\frac{1}{1+(\omega-2)^2} - \frac{1}{1+(\omega+2)^2} \right] = \frac{i}{\sqrt{2\pi}} \left[\frac{1}{1+(\omega+2)^2} - \frac{1}{1+(\omega-2)^2} \right]$$

28 If we denote $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

then
$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$

Now consider $g(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}$

we can write $g(x) = p\left(x - \frac{a+b}{2}\right)$,
$$p(x) = f\left(\frac{x}{\frac{b-a}{2}}\right)$$

$$\underset{\substack{\uparrow \\ \text{shifting (Exer 19)}}}{\hat{g}(\omega) = e^{-i\left(\frac{a+b}{2}\right)\omega} \hat{p}(\omega)} \underset{\substack{\uparrow \\ \text{dilation}}}{=} e^{-i\left(\frac{a+b}{2}\right)\omega} \hat{f}\left(\frac{b-a}{2}\omega\right) \cdot \left(\frac{b-a}{2}\right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{b-a}{2}\right) e^{-i\frac{a+b}{2}\omega} \frac{\sin\left(\frac{b-a}{2}\omega\right)}{\frac{b-a}{2}\omega}$$

$$= \sqrt{\frac{2}{\pi}} e^{-i\frac{a+b}{2}\omega} \frac{\sin\left(\frac{b-a}{2}\omega\right)}{\omega}$$

the ~~past~~ function plotted is
$$h\, f\left(\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}}\right) + k\, f\left(\frac{x - \frac{c+d}{2}}{\frac{d-c}{2}}\right)$$

and the Fourier transform is
$$\sqrt{\frac{2}{\pi}} \left[e^{-i\frac{a+b}{2}\omega} \frac{\sin\left(\frac{b-a}{2}\omega\right)}{\omega} + e^{-i\left(\frac{c+d}{2}\right)\omega} \frac{\sin\left(\frac{d-c}{2}\omega\right)}{\omega} \right]$$

40. Let $g(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$ then $f(x) = x\,g(x)$

$$\mathcal{F}(xg(x)) \underset{\substack{\uparrow \\ \text{Theorem 3}}}{=} i\frac{d}{d\omega}\mathcal{F}(g)(\omega) \underset{\substack{\uparrow \\ \text{Example 2}}}{=} i\frac{d}{d\omega}\left[\frac{1-i\omega}{\sqrt{2\pi}\,(1+\omega^2)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{-\omega^2-2i\omega+1}{(1+\omega^2)^2} = -\frac{1}{\sqrt{2\pi}}\,\frac{(i+\omega)^2}{(1+\omega^2)^2}$$

Section 7.3

2. $$\frac{d^2}{dt^2}\hat{u} = -\omega^2\hat{u} \qquad \hat{u} = A(\omega)\cos\omega t + B(\omega)\sin\omega t$$

Since $g(x) = 0$, $f(x) = \begin{cases} \cos x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

$$\hat{g}(\omega) = 0, \quad \hat{f}(\omega) = (-i\omega)\,i\sqrt{\frac{2}{\pi}}\,\frac{\omega\cos\left(\frac{\omega\pi}{2}\right)}{\omega^2-1}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\omega^2\cos\left(\frac{\pi\omega}{2}\right)}{\omega^2-1}$$

So $$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega^2\cos\left(\frac{\pi\omega}{2}\right)}{\omega^2-1}\cos\omega t\; e^{i\omega x}\,d\omega$$

4. $$\hat{f}(\omega) = 100\cdot\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}100\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}\,e^{-\frac{\omega^2 t}{100}}\,e^{i\omega x}\,d\omega$$

$$= \frac{100}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}\,e^{-\frac{\omega^2 t}{100}+i\omega x}\,d\omega$$

10. $$a\cdot(i\omega)\,\hat{u} + \frac{d\hat{u}}{dt} = 0$$

$$\hat{u} = \hat{f}(\omega)\,\cancel{e^{-ia(t)\omega}}\;e^{-i\omega\int_0^t a(s)\,ds}$$

$\cancel{u(x,t) =}$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega \int_0^t a(s)\,ds} \cdot e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega \left(x - \int_0^t a(s)\,ds\right)}\, d\omega$$

$$= f\left(x - \int_0^t a(s)\,ds\right)$$

15. $$\frac{d^2\hat{u}}{dt^2} + 2\frac{d\hat{u}}{dt} = -\hat{u}$$

$$\hat{u} = A(\omega)\, e^{-t} + B(\omega)\, t e^{-t}$$

$$\frac{d\hat{u}}{dt} = -A(\omega)\, e^{-t} + B(\omega)\left[e^{-t} - t e^{-t}\right]$$

at $t = 0$ $\quad \hat{u} = A(\omega) = \hat{f}(\omega)$

$$\frac{d\hat{u}}{dt} = -A(\omega) + B(\omega) = \hat{g}(\omega)$$

therefore $\quad \hat{u} = \hat{f}(\omega)\, e^{-t} + \left(\hat{g}(\omega) + \hat{f}(\omega)\right) t e^{-t}$

$$u(x,t) = f(x)\, e^{-t} + \left(f(x) + g(x)\right) t e^{-t}$$

23 $$\frac{d\hat{u}}{dt} = c^2(-\omega^2)\,\hat{u} + k i \omega\, \hat{u} = \left(ik\omega - c^2\omega^2\right)\hat{u}$$

$$\hat{u} = \hat{f}(\omega)\, e^{(ik\omega - c^2\omega^2)t}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{(ik\omega - c^2\omega^2)t} \cdot e^{i\omega x}\, d\omega$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i(kt+x)\omega}}\ \cdots$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i(kt+x)\omega - c^2\omega^2 t}\, d\omega$$