Section 3.5

4. $$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} 100 \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} 0 \, dx$$

$$= \frac{200}{\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi/2} = \frac{200}{n\pi} \left[1 - \cos(n\pi/2) \right]$$

$$= \begin{cases} \dfrac{200}{n\pi} & n \text{ odd} \\ \dfrac{400}{n\pi} & \dfrac{n}{2} \text{ odd} \\ 0 & \dfrac{n}{2} \text{ even} \end{cases}$$

6. $$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$$

$$b_n = 2 \int_0^1 e^{-x} \sin n\pi x \, dx$$

from the appendixes

$$= 2 \left[\frac{e^{-x}}{1+n^2\pi^2} \left(-\sin n\pi x - n\pi \cos n\pi x \right) \right]_0^1$$

$$= 2 \left[\frac{e^{-1}}{1+n^2\pi^2} (0 - n\pi \cos n\pi) - \frac{e^0}{1+n^2\pi^2} (0 - n\pi) \right]$$

$$= \frac{2}{1+n^2\pi^2} \left[\frac{1}{e} n\pi \cdot (-1)^{n+1} + n\pi \right] = \frac{2n\pi}{1+n^2\pi^2} \left(1 - \frac{(-1)^n}{e} \right)$$

10. (b) We just need to verify

$$\frac{2}{L} \int_0^L \left(T_1 + \frac{T_2 - T_1}{L} x \right) \sin \frac{n\pi}{L} x \, dx = 2 \, \frac{T_1 + (-1)^{n+1} T_2}{n\pi}$$

12. $b_n = 2\int_0^1 50x(1-x)\sin n\pi x\,dx - 2\int_0^1 100\sin n\pi x\,dx$

$= 100\int_0^1 x\sin n\pi x\,dx - 100\int_0^1 x^2\sin n\pi x\,dx - 200\int_0^1 \sin n\pi x\,dx$

$= 100\cdot\frac{2}{(n\pi)^3}\left[1-(-1)^n\right] - \frac{200}{n\pi}\left[1-(-1)^n\right]$

$= \frac{200}{n\pi}\left[1-(-1)^n\right]\cdot\left(\frac{1}{(n\pi)^2}-1\right)$

Section 3.6

2. $u(x,t) = e^{-\pi^2 t}\cos\pi x$

7. Integrating the heat equation from 0 to L

$$\frac{d}{dt}\int_0^L u\,dx = c^2\int_0^L \frac{\partial^2 u}{\partial x^2}dx = c^2\left[\frac{\partial u}{\partial x}\right]_0^L$$

Since $\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0$, (the bar is insulated)

we have $\frac{d}{dt}\int_0^L u\,dx = 0$

or $\int_0^L u(x,t)\,dx = \text{const}$

14. $u = \sum_{n=1}^{\infty} c_n e^{-c^2\mu_n^2 t}\sin\mu_n x = \sum_{n=1}^{\infty} c_n e^{-\mu_n^2 t}\sin\mu_n x$

$c_n = \int_0^1 f(x)\sin\mu_n x\,dx = \int_{1/2}^1 100(x-\tfrac{1}{2})\sin\mu_n x\,dx$

$= \cancel{-\frac{1}{\mu_n}\cdot\frac{1}{2}\cos\frac{\mu_n}{2} + \frac{1}{\mu_n^2}\sin\frac{\mu_n}{2}}$

$= -\frac{1}{\mu_n}\cdot\frac{1}{2}\cos\mu_n + \frac{1}{\mu_n^2}\left(\sin\mu_n - \sin\frac{\mu_n}{2}\right) \qquad n = 1, 2, \dots$