Section 3.5

13 Steady state solution : $u_1(x) = \frac{50-100}{\pi} x + 100 = 100 - 50x/\pi$

$$f(x) - \left(100 - \frac{50x}{\pi}\right) = \begin{cases} \left(33 + \frac{50}{\pi}\right)x - 100 & 0 \le x \le \pi/2 \\ 33\pi - 100 - \left(33 + \frac{50}{\pi}\right)x, & \pi/2 < x < \pi \end{cases}$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \left[\left(33 + \frac{50}{\pi}\right)x - 100\right] \sin nx\, dx + \int_{\pi/2}^{\pi} \left[33\pi - 100 - \left(33 + \frac{50}{\pi}\right)x\right] \sin nx\, dx \right.$$

$$u(x,t) = 100 - 50\frac{x}{\pi} + \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

Section 3.6

3. $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi/2} 100x\, dx + \int_{\pi/2}^{\pi} 100(\pi - x)\, dx\right]$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} 100x \cos nx\, dx + \int_{\pi/2}^{\pi} 100(\pi - x) \cos nx\, dx\right]$$

12. $c_n = \int_0^1 \sin \pi x \sin \mu_n x\, dx \Big/ \int_0^1 \sin^2 \mu_n x\, dx$

$$\int_0^1 \sin^2 \mu_n x\, dx = \int_0^1 \frac{1 - \cos 2\mu_n x}{2} dx = \frac{1}{2} - \frac{1}{4\mu_n} \sin 2\mu_n$$

$$\int_0^1 \sin \pi x \sin \mu_n x\, dx = \frac{1}{2(\pi - \mu_n)} \sin(\pi - \mu_n) - \frac{1}{2(\pi + \mu_n)} \sin(\pi + \mu_n)$$

$c_1 \approx 0.8193$, $c_2 \approx 0.4150$, ---

Section 3.7

2. $$B_{mn} = 4 \int_0^{\pi} \int_0^{\pi} \sin \pi x \sin \pi y \sin m\pi x \sin n\pi y\, dx\, dy$$
$$= \left[2 \int_0^{\pi} \sin \pi x \sin m\pi x\, dx\right] \cdot \left[2 \int_0^{\pi} \sin \pi y \sin n\pi y\, dy\right]$$

$$= \delta_{m,1} \cdot \delta_{n,1} = \begin{cases} 1 & \text{if } m=1 \text{ and } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$B^*_{mn} = \frac{4}{\sqrt{m^2+n^2}} \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin n\pi y \, dx \, dy$$

$$= \frac{4}{\sqrt{m^2+n^2}} \int_0^1 \sin \pi x \sin m\pi x \, dx \cdot \int_0^1 \sin n\pi y \, dy$$

$$= \frac{2}{\sqrt{m^2+n^2}} \cdot \delta_{m,1} \cdot \frac{1}{n\pi} (1 - \cos n\pi)$$

$$= \begin{cases} \frac{2}{\sqrt{1+n^2}} \cdot \frac{1}{n\pi} (1 - (-1)^n) & m=1 \\ 0 & m \neq 1 \end{cases}$$

$$u(x,y,t) = \cos\sqrt{2}t \sin\pi x \sin\pi y + \sum_{n=1}^{\infty} \cancel{} \frac{2}{\sqrt{1+n^2}} \frac{1}{n\pi} (1-(-1)^n) \cdot$$

$$\cdot \left[\sin\sqrt{n^2+1}\, t \sin\pi x \sin n\pi y \right]$$

$$= \cos\sqrt{2}t \sin\pi x \sin\pi y + \sum_{k=0}^{\infty} \frac{4 \sin\sqrt{1+(2k+1)^2}\, t}{(2k+1)\pi \sqrt{1+(2k+1)^2}} \sin\pi x \sin(2k+1)\pi y$$

6. $B_{mn} = 0$

$$B^*_{mn} = \frac{4}{\sqrt{m^2+n^2}} \int_0^1 x \sin m\pi x \, dx \cdot \int_0^1 (1-y) \sin n\pi y \, dy$$

10. See Matlab output

Section 3.8

4 $u = u_1(x,y) + u_2(x,y)$

$$u_1 = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi(1-y)$$

$$A_n = \frac{2}{\sinh n\pi} \int_0^1 (1-x) \sin n\pi x \, dx = \frac{2}{n\pi \sinh n\pi}$$

$$u_2 = \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

$$B_n = \frac{2}{\cancel{\sinh n\pi}\ \sinh n\pi} \int_0^1 x \sin n\pi x \, dx = \frac{2(-1)^{n+1}}{n\pi \sinh n\pi}$$

$$u(x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \sinh n\pi} \left[\sinh n\pi(1-y) + (-1)^{n+1} \sinh n\pi y \right] \sin n\pi x$$

6 Section 3.9

14 (a) Related homogeneous problem $\nabla^2 u = 0$, $u=0$ on all six sides

(b) Try $\varphi_{lmn}(x,y,z) = \sin \frac{l\pi}{a} x \sin \frac{m\pi}{b} y \sin \frac{n\pi}{c} z$

$$\nabla^2 \varphi_{lmn} = -\left[\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 \right] \varphi_{lmn}$$

so $\lambda_{lmn} = +\pi^2 \left[\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{n}{c}\right)^2 \right]$ $(l, m, n = 1, 2, \dots)$

(c) Suppose that $f(x,y,z)$ is defined for all $0<x<a$, $0<y<b$, and $0<z<c$. Then we have the tripple Fourier series expansion

$$f(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{lmn} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}$$

where $B_{lmn} = \frac{8}{abc} \int_0^c \int_0^b \int_0^a f(x,y,z) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \, dx\, dy\, dz$

Now we can solve the Poisson problem in a box by looking for solution in the form

$$u(x,y,z) = \sum_{l}\sum_{m}\sum_{n} E_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi x}{b} \sin\frac{n\pi z}{c}$$

This satisfies the homogeneous boundary conditions automatically

$$\nabla^2 u = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-E_{lmn}\lambda_{lmn}\right]\varphi_{lmn} = \sum_{l}\sum_{m}\sum_{n} B_{lmn}\varphi_{lmn}$$

By setting coefficients to match

$$E_{lmn} = -\frac{B_{lmn}}{\lambda_{lmn}}$$

Section 4.1

3. $u(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{r}$, $\left(\dfrac{1}{r}\right)' = -\dfrac{1}{r^2}$, $\left(\dfrac{1}{r}\right)'' = \dfrac{2}{r^3}$

$$\nabla^2 u = \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{1}{r}\left(-\frac{1}{r^2}\right) = \frac{1}{r^3} \neq 0$$

6. $u(x,y) = \log(x^2+y^2) = \log r^2 = 2\log r$

$(\log r)' = \dfrac{1}{r}$, $(\log r)'' = -\dfrac{1}{r^2}$

$$\nabla^2 u = (\log r)'' + \frac{1}{r}(\log r)' = -\frac{1}{r^2} + \frac{1}{r}\cdot\frac{1}{r} = 0$$