Section 2.3

2. $f(x) = x$ if $-p < x < p$

(a) This function is odd.

(b) Compare with Exercise 13. Section 2.2

$$x \longrightarrow x' = \frac{p}{\pi} x \qquad -\pi < x < \pi \Rightarrow -p < x' < p$$

if $w(x) = x$, $-\pi < x < \pi \Rightarrow w(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x'}{p} = \frac{\pi}{p} x'$$

This holds for $-\pi < x < \pi$, or $-p < x' < p$:

$$x' = \frac{p}{\pi} \cdot 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x'}{p}$$

change notation back to x :

$$f(x) = x = \frac{2p}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{p} \qquad -p < x < p$$

Discontinuities $\quad x = \pm p, \pm 3p, \ldots$

where $\frac{1}{2}[f(p_+) + f(p_-)] = 0$, or

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi \frac{p}{p}) = 0$$

6. $$f(x) = \begin{cases} c & \text{if } |x| < d \\ 0 & \text{if } d < |x| < p \end{cases} \qquad \text{where } 0 < d < p$$

$f(x)$ is even, so $b_n = 0$

$$a_0 = c \cdot \frac{d}{p}$$

$$a_n = \frac{2}{p}\int_0^p f(x) \cos\frac{n\pi}{p} x\, dx = \frac{2}{p}\int_0^d c \cos\frac{n\pi}{p} x\, dx$$

$$= \frac{2c}{p} \frac{p}{n\pi} \sin\frac{n\pi x}{p}\Big|_0^d = \frac{2c}{n\pi} \sin\frac{n\pi d}{p}$$

$$f(x) = \frac{cd}{p} + \frac{2c}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi d}{p}}{n} \cos\frac{n\pi x}{p}$$

20. (a) $f_e(-x) = \dfrac{f(-x) + f(-(-x))}{2} = \dfrac{f(-x) + f(x)}{2} = f_e(x)$

$f_o(-x) = \dfrac{f(-x) - f(-(-x))}{2} = \dfrac{f(-x) - f(x)}{2} = -\dfrac{f(x) - f(-x)}{2} = -f_o(x)$

(b) $f_e(x) + f_o(x) = \dfrac{f(x) + f(-x)}{2} + \dfrac{f(x) - f(-x)}{2} = f(x)$

If $f(x) = g_e(x) + g_o(x)$ where g_e is even and g_o is odd

then $g_e(x) + g_o(x) = f_e(x) + f_o(x)$, for all x.

We would like to show that $f_e(x) = g_e(x)$ and $f_o(x) = g_o(x)$

so the decomposition is unique. By moving terms

$$\underbrace{g_e(x) - f_e(x)}_{\text{still even}} = \underbrace{f_o(x) - g_o(x)}_{\text{still odd}}$$

The only function that is even and odd at the same time is ~~zero~~ ! So we must have

$$g_e(x) = f_e(x) \quad \text{and} \quad f_o(x) = g_o(x)$$

The decomposition is therefore unique!

(c) $f(x)$ $2p$-periodic $\rightarrow$ $f(-x)$ $2p$-periodic

$\Rightarrow$ f_e and f_o are also (sum, difference of two periodic functions)

(d) $$f(x) = a_0 + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p}}_{\substack{\text{even} \\ \| \\ f_e(x)}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}}_{\substack{\text{odd} \\ \| \\ f_o(x)}}$$

by uniqueness of the decomposition

Section 2.4

2 $f(x) = \pi - x$ if $0 \le x \le \pi$

(a) sine expansion

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx\,dx \quad , \quad n = 1, 2, \cdots$$

(b) cosine expansion

$$a_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx \quad , \quad n = 1, 2, \cdots$$

6. $f(x) = \cos x$ if $0 < x < \pi$

(a) cosine expansion

$$f(x) = \cos x$$

(b) sine expansion

$$b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx = \begin{cases} 0 & n = 1 \\ A_n & n \ne 1 \end{cases}$$

$$A_n = \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{1}{n+1}\left(1 - \cos(n+1)\pi\right) + \frac{1}{2}\,\frac{1}{n-1}\left(1 - \cos(n-1)\pi\right)\right]$$

$$= \begin{cases} 0 & n \text{ odd} \\ \frac{1}{n+1} + \frac{1}{n-1} & n \text{ even} \end{cases}$$

Section 3.1

2 linear in equation and boundary conditions, nonhomogeneous equation

Section 3.3

4. $u(x,t) = \sin \pi x \cos \pi t + \sin 2\pi x \cdot \frac{1}{2\pi}\sin 2\pi t + \frac{1}{2}\sin 3\pi x \cos 3\pi t$
$+ 3\sin 7\pi x \cos 7\pi t$

14. Compare to Exercise 12, $k = 1/2$, $c = 1$, $L = \pi$

$\frac{kL}{\pi c} = \frac{1}{2}$ is not a positive integer so the solution is

$$u(x,t) = e^{-\frac{1}{2}t} \sum_{n > 1/2} \sin \frac{n\pi}{\pi} x \,(a_n \cos \lambda_n t + b_n \sin \lambda_n t)$$

$$a_n = \frac{2}{\pi} \int_0^\pi x \sin x \cdot \sin \frac{n\pi}{L} x \, dx = \frac{2}{\pi} \int_0^\pi x \sin x \sin nx \, dx$$

b_n are determined from

$$-\frac{1}{2} a_n + \lambda_n b_n = 0 \Rightarrow b_n = \frac{a_n}{2\lambda_n}$$

The integral $\int_0^\pi x \sin x \sin nx \, dx$ can be computed by maple or following formulas in the table :

Hint :

$$x \sin x \sin nx = x \left[\frac{1}{2} \frac{\sin (n-1)x}{n-1} - \frac{1}{2} \frac{\sin (n+1)x}{n+1} \right]$$

if $x \neq 1$

If $n = 1$ $\quad x \sin^2 x = x \cdot \frac{1 - \cos 2x}{2}$

The integrals of these two functions on the right can certainly be found using formulas in the table.