

Systems of Differential Equations and Laplace's Method

- Solving $x' = Cx$
- The Resolvent
- An Illustration for $x' = Cx$
- Solving $x'' = Ax$

Solving $x' = Cx$

Apply L to each side to obtain $L(x') = CL(x)$. Use the parts rule

$$L(x') = sL(x) - x(0)$$

to obtain

$$\begin{aligned} sL(x) - x(0) &= L(Cx) \\ sL(x) - L(Cx) &= x(0) \\ sI L(x) - C L(x) &= x(0) \\ (sI - C)L(x) &= x(0). \end{aligned}$$

Resolvent

The inverse of $sI - C$ is called the **resolvent**, a term invented to describe the equation

$$L(x(t)) = (sI - C)^{-1}x(0).$$

An Illustration for $\mathbf{x}' = C\mathbf{x}$

Define $C = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives a scalar initial value problem

$$\begin{cases} x_1'(t) = 2x_1(t) + 3x_2(t), \\ x_2'(t) = 4x_2(t), \\ x_1(0) = 1, \\ x_2(0) = 2. \end{cases}$$

Then the adjugate formula $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$ gives the resolvent

$$(sI - C)^{-1} = \frac{1}{(s-2)(s-4)} \begin{pmatrix} s-4 & -3 \\ 0 & s-2 \end{pmatrix}.$$

The Laplace transform of the solution is then

$$L(\mathbf{x}(t)) = (sI - C)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{s - 10}{(s - 2)(s - 4)} \\ \frac{2}{s - 4} \end{pmatrix}.$$

Partial fractions and use of the backward Laplace table imply

$$\mathbf{x}(t) = \begin{pmatrix} 4e^{2t} - 3e^{4t} \\ 2e^{4t} \end{pmatrix}.$$

Laplace Operations on $x'' = Ax$

Apply L to each side to obtain $L(x'') = AL(x)$. Use the parts rule to obtain

$$\begin{aligned}L(x'') &= s(sL(x) - x(0)) - x'(0) \\s^2L(x) - sx(0) - x'(0) &= AL(x) \\(s^2I - A)L(x) &= sx(0) + x'(0) \\L(x(t)) &= (s^2I - A)^{-1} (sx(0) + x'(0)).\end{aligned}$$

Resolvent Formula

The inverse of $s^2I - C$ is also called a **resolvent**, and we have the equation

$$L(\mathbf{x}(t)) = (s^2I - A)^{-1} (s\mathbf{x}(0) + \mathbf{x}'(0)).$$

Formal Solution of $\mathbf{x}'' = A\mathbf{x}$

$$\mathbf{x}(t) = L^{-1} \left((s^2I - A)^{-1} (s\mathbf{x}(0) + \mathbf{x}'(0)) \right).$$