The Integrating Factor Method for a Linear Differential Equation $y' + p(x)y = r(x)$

Contents

Superposition

Consider the homogeneous equation

$$y' + p(x)y = 0 \tag{1}$$

and the non-homogeneous equation

$$y' + p(x)y = r(x) \tag{2}$$

where p and r are continuous in an interval J.

Theorem 1 (Superposition)

The general solution of the non-homogeneous equation (2) is given by

$$y = y_h + y_p$$

where y_h is the general solution of homogeneous equation (1) and y_p is a particular solution of non-homogeneous equation (2).

Variation of Parameters

The initial value problem

$$y' + p(x)y = r(x), \quad y(x_0) = 0, \tag{3}$$

where p and r are continuous in an interval containing $x = x_0$, has a particular solution

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t) e^{\int_{x_0}^{t} p(s)ds}\, dt. \tag{4}$$

Formula (4) is called **variation of parameters**, for historical reasons.

The formula determines a particular solution y_p which can be used in the superposition identity $y = y_h + y_p$.

While (4) has some appeal, applications use the **integrating factor method**, which is developed with indefinite integrals for computational efficiency. No one memorizes (4); they remember and study the *method*.

Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule

$$\text{(5)} \qquad \text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}.$$

The factor $W = e^{\int p(x)dx}$ in (5) is called an **integrating factor**.

Details

Let $W = e^{\int p(x)dx}$. Then $W' = pW$, by the rule $(e^x)' = e^x$, the chain rule and the fundamental theorem of calculus $(\int p(x)dx)' = p(x)$.

Let's prove $(WY)'/W = Y' + pY$. The derivative product rule implies

$$\begin{aligned} (YW)' &= Y'W + YW' \\ &= Y'W + YpW \\ &= (Y' + pY)W. \end{aligned}$$

The proof is complete.

The Integrating Factor Method

Standard Form Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.

Find W Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.

Prepare for Quadrature Obtain the new equation $\dfrac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (5).

Method of Quadrature Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

Equation (5) is central to the method, because it collapses the two terms $y' + py$ into a single term $(yW)'/W$; the method of quadrature applies to $(yW)' = rW$. Literature calls the exponential factor W an **integrating factor** and equivalence (5) a **factorization** of $Y' + p(x)Y$.

Integrating Factor Example

Example. Solve the linear differential equation $xy' + y = x^2$.

Solution: The standard form of the linear equation is

$$y' + \frac{1}{x}y = x.$$

Let

$$W = e^{\int \frac{1}{x} dx} = x$$

and replace the LHS of the differential equation by $(yW)'/W$ to obtain the quadrature equation

$$(yW)' = xW.$$

Apply quadrature to this equation, then divide by W, to give the answer

$$y = \frac{x^2}{3} + \frac{C}{x}.$$