Fourier Series

- Fourier Sine Series
- Fourier Cosine Series
- Fourier Series
  - Convergence of Fourier Series for $2T$-Periodic Functions
  - Convergence of Half-Range Expansions: Cosine Series
  - Convergence of Half-Range Expansions: Sine Series
- Sawtooth Wave
- Triangular Wave
- Parseval's Identity and Bessel's Inequality
- Complex Fourier Series
- Dirichlet Kernel and Convergence

Fourier Sine Series

Definition. Consider the orthogonal system $\left\{\sin\left(\frac{n\pi x}{T}\right)\right\}_{n=1}^{\infty}$ on $[-T, T]$. A Fourier sine series with coefficients $\{b_n\}_{n=1}^{\infty}$ is the expression

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right)$$

Theorem. A Fourier sine series $F(x)$ is an odd $2T$-periodic function.

Theorem. The coefficients $\{b_n\}_{n=1}^{\infty}$ in a Fourier sine series $F(x)$ are determined by the formulas (inner product on $[-T, T]$)

$$b_n = \frac{\left\langle F, \sin\left(\frac{n\pi x}{T}\right)\right\rangle}{\left\langle \sin\left(\frac{n\pi x}{T}\right), \sin\left(\frac{n\pi x}{T}\right)\right\rangle} = \frac{2}{T}\int_0^T F(x)\sin\left(\frac{n\pi x}{T}\right) dx.$$

Fourier Cosine Series

Definition. Consider the orthogonal system $\{\cos\left(\frac{m\pi x}{T}\right)\}_{m=0}^{\infty}$ on $[-T, T]$. A Fourier cosine series with coefficients $\{a_m\}_{m=0}^{\infty}$ is the expression

$$F(x) = \sum_{m=0}^{\infty} a_m \cos\left(\frac{m\pi x}{T}\right)$$

Theorem. A Fourier cosine series $F(x)$ is an even $2T$-periodic function.

Theorem. The coefficients $\{a_m\}_{m=0}^{\infty}$ in a Fourier cosine series $F(x)$ are determined by the formulas (inner product on $[-T, T]$)

$$a_m = \frac{\left\langle F, \cos\left(\frac{m\pi x}{T}\right)\right\rangle}{\left\langle \cos\left(\frac{m\pi x}{T}\right), \cos\left(\frac{m\pi x}{T}\right)\right\rangle} = \begin{cases} \frac{2}{T}\int_0^T F(x)\cos\left(\frac{m\pi x}{T}\right)dx & m > 0, \\ \frac{1}{T}\int_0^T F(x)\,dx & m = 0. \end{cases}$$

Fourier Series

Definition. Consider the orthogonal system $\left\{\cos\left(\frac{m\pi x}{T}\right)\right\}_{m=0}^{\infty}, \left\{\sin\left(\frac{n\pi x}{T}\right)\right\}_{n=1}^{\infty}$, on $[-T, T]$. A Fourier series with coefficients $\{a_m\}_{m=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ is the expression

$$F(x) = \sum_{m=0}^{\infty} a_m \cos\left(\frac{m\pi x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right)$$

Theorem. A Fourier series $F(x)$ is a $2T$-periodic function.

Theorem. The coefficients $\{a_m\}_{m=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ in a Fourier series $F(x)$ are determined by the formulas (inner product on $[-T, T]$)

$$a_m = \frac{\left\langle F, \cos\left(\frac{m\pi x}{T}\right)\right\rangle}{\left\langle \cos\left(\frac{m\pi x}{T}\right), \cos\left(\frac{m\pi x}{T}\right)\right\rangle} = \begin{cases} \frac{1}{T}\int_{-T}^{T} F(x)\cos\left(\frac{m\pi x}{T}\right) dx & m > 0, \\ \frac{1}{2T}\int_{-T}^{T} F(x)\, dx & m = 0. \end{cases}$$

$$b_n = \frac{\left\langle F, \sin\left(\frac{n\pi x}{T}\right)\right\rangle}{\left\langle \sin\left(\frac{n\pi x}{T}\right), \sin\left(\frac{n\pi x}{T}\right)\right\rangle} = \frac{1}{T}\int_{-T}^{T} F(x)\sin\left(\frac{n\pi x}{T}\right) dx.$$

Convergence of Fourier Series for $2T$-Periodic Functions

The Fourier series of a $2T$-periodic piecewise smooth function $f(x)$ is

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \right)$$

where

$$a_0 = \frac{1}{2T} \int_{-T}^{T} f(x) dx,$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx,$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right) dx.$$

The series converges to $f(x)$ at points of continuity of f and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Convergence of Half-Range Expansions: Cosine Series

The Fourier cosine series of a piecewise smooth function $f(x)$ on $[0, T]$ is the even $2T$-periodic function

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right)$$

where

$$a_0 = \frac{1}{T}\int_0^T f(x)dx,$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{n\pi x}{T}\right)dx.$$

The series converges on $0 < x < T$ to $f(x)$ at points of continuity of f and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Convergence of Half-Range Expansions: Sine Series

The Fourier sine series of a piecewise smooth function $f(x)$ on $[0, T]$ is the odd $2T$-periodic function

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right)$$

where

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi x}{T}\right)dx.$$

The series converges on $0 < x < T$ to $f(x)$ at points of continuity of f and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Sawtooth Wave

Definition. The **sawtooth wave** is the odd 2π-periodic function defined on $-\pi \leq x \leq \pi$ by the formula

$$\mathrm{sawtooth}(x) = \begin{cases} \frac{1}{2}(\pi - x) & 0 < x \leq \pi, \\ \frac{1}{2}(-\pi - x) & -\pi \leq x < 0, \\ 0 & x = 0. \end{cases}$$

Theorem. The sawtooth wave has Fourier sine series

$$\mathrm{sawtooth}(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

Triangular Wave

Definition. The **triangular wave** is the even 2π-periodic function defined on $-\pi \leq x \leq \pi$ by the formula

$$\mathrm{twave}(x) = \begin{cases} \pi - x & 0 < x \leq \pi, \\ \pi + x & -\pi \leq x \leq 0. \end{cases}$$

Theorem. The triangular wave has Fourier cosine series

$$\mathrm{twave}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x.$$

ParsevaL's Identity and Bessel's Inequality

Theorem. (Bessel's Inequality)

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \leq \frac{1}{2T}\int_{-T}^{T}|f(x)|^2 dx$$

Theorem. (Parseval's Identity)

$$\frac{1}{2T}\int_{-T}^{T}|f(x)|^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Theorem. Parseval's identity for the sawtooth function implies

$$\frac{\pi^2}{12} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Complex Fourier Series

Definition. Let $f(x)$ be $2T$-periodic and piecewise smooth. The complex Fourier series of f is

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}}, \quad c_n = \frac{1}{2T}\int_{-T}^{T} f(x) e^{\frac{-in\pi x}{T}} dx$$

Theorem. The complex series converges to $f(x)$ at points of continuity of f and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Theorem. (Complex Parseval Identity)

$$\frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Dirichlet Kernel and Convergence

Theorem. (Dirichlet Kernel Identity)

$$\frac{1}{2} + \cos u + \cos 2u + \cdots + \cos nu = \frac{\sin\left(\left(n + \frac{1}{2}\right) u\right)}{2 \sin\left(\frac{1}{2} u\right)}$$

Theorem. (Riemann-Lebesgue)

For piecewise continuous $g(x)$, $\lim_{N \to \infty} \int_{-\pi}^{\pi} g(x) \sin(Nx) dx = 0$.

Proof: Integration theory implies it suffices to establish the result for smooth g. Integrate by parts to obtain $\frac{1}{n}(g(-\pi) - g(\pi))(-1)^n + \frac{1}{n} \int_{-\pi}^{\pi} g(x) \cos(nx) dx$. Letting $n \to \infty$ implies the result.

Theorem. Let $f(x)$ be 2π-periodic and smooth on the whole real line. Then the Fourier series of $f(x)$ converges uniformly to $f(x)$.

Convergence Proof

STEP 1. Let $s_N(x)$ denote the Fourier series partial sum. Using Dirichlet's kernel formula, we verify the identity

$$f(x) - s_N(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} (f(x) - f(x+w)) \left(\frac{\sin((N+1/2)w)}{2\sin(w/2)}\right) dw$$

STEP 2. The integrand I is re-written as

$$I = \frac{f(x) - f(x+w)}{w}\frac{w}{2\sin(w/2)}\sin((N+1/2)w).$$

STEP 3. The function $g(w) = \dfrac{f(x) - f(x+w)}{w}\dfrac{w}{\sin(w/2)}$ is piecewise continuous. Apply the Riemann-Lebesgue Theorem to complete the proof of the theorem.

Gibbs' Phenomena

Engineering Interpretation: The graph of $f(x)$ and the graph of

$$a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$$

are identical to pixel resolution, provided N is sufficiently large. Computers can therefore graph $f(x)$ using a truncated Fourier series.

If $f(x)$ is only piecewise smooth, then pointwise convergence is still true, at points of continuity of f, but uniformity of the convergence fails near discontinuities of f and f'. Gibbs discovered the fixed-jump artifact, which appears at discontinuities of f.