Pure and Practical Resonance in Forced Vibrations

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Definition of Pure Resonance

The notion of **pure resonance** in the differential equation

$$x''(t) + \omega_0^2 x(t) = F_0 \cos(\omega t) \tag{1}$$

is the existence of a solution that is unbounded as $t \to \infty$. We already know that for $\omega \neq \omega_0$, the general solution of (1) is the sum of two harmonic oscillations, hence it is bounded.

Equation (1) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution

$$x(t) = \frac{F_0}{2\omega_0} t \sin(\omega_0 t).$$

Pure resonance occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω, in which case all solutions of the differential equation are unbounded.

Typical Pure Resonance Graphic

In Figure 1, pure resonance is illustrated for $x''(t) + 16x(t) = 8\cos 4t$, which in (1) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$.

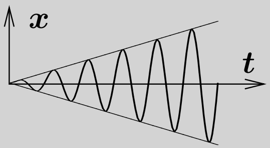

Figure 1. Pure resonance.
Graphed are the envelope curves $x = \pm t$ and the solution $x(t) = t\sin 4t$ of the equation $x''(t) + 16x(t) = 8\cos\omega t$, where $\omega = 4$.

Pure Resonance Explained by Undetermined Coefficients

An initial trial solution of

$$x''(t) + 16x(t) = 8\cos\omega t$$

is

$$x = d_1\cos\omega t + d_2\sin\omega t.$$

The homogeneous solution $x_h = c_1\cos 4t + c_2\sin 4t$ considered in the **correction rule** has duplicate terms exactly when the natural frequencies match: $\omega = 4$. Then the final trial solution is classified as follows.

$$\begin{array}{llll} \omega \ne 4 & \Longrightarrow & x(t) = d_1\cos\omega t + d_2\sin\omega t & \textbf{is bounded, no resonance,} \\ \omega = 4 & \Longrightarrow & x(t) = t(d_1\cos\omega t + d_2\sin\omega t) & \textbf{is unbounded, pure resonance.} \end{array}$$

Even before the undetermined coefficients d_1, d_2 are evaluated, we can decide that unbounded solutions occur exactly when frequency matching $\omega = 4$ occurs, because of the amplitude factor t. If $\omega \ne 4$, then $x_p(t)$ is a pure harmonic oscillation, which implies it is bounded. If $\omega = 4$, then $x_p(t)$ equals a time–varying amplitude Ct times a pure harmonic oscillation, hence it is unbounded.

The Wine Glass Experiment

Equation $x''(t) + \omega_0^2\,x(t) = F_0\cos(\omega t)$ is advertised as the basis for a physics experiment in the Public Television Annenberg CPB Project, called the *wine glass experiment*.

Cal Tech physicist Goodstein, in front of an audience of physics students, equips a lab table with a frequency generator, an amplifier and an audio speaker. The *valuable* wine glass is replaced by a glass lab beaker. The frequency generator is tuned to the natural frequency of the glass beaker ($\omega \approx \omega_0$), then the volume knob on the amplifier is suddenly turned up (F_0 adjusted larger), whereupon the sound waves emitted from the speaker break the glass beaker.

The 29-minute CPB video can be viewed in Windows Media Player, no cost, in *Waves* #17 at the link below. The wine glass experiment is the last 4 minutes of the film.

`http://www.free-ed.net/free-ed/Science/Physics/`

The glass itself will vibrate at a certain frequency, as can be determined experimentally by *pinging* the glass rim. This vibration operates within elastic limits of the glass and the glass will not break under these circumstances. A physical explanation for the breakage is that an incoming sound wave from the speaker is timed to add to the glass rim excursion. After enough amplitude additions [from 1mm initially to eventually 5mm], the glass rim moves beyond the elastic limit and the glass breaks.

The explanation implies that the external frequency from the speaker has to match the natural frequency of the glass. But there is more to it: the glass has some natural damping that nullifies feeble attempts to increase the glass rim amplitude. The physicist uses to great advantage this natural damping to *tune* the external frequency to the glass. The reason for turning up the volume on the amplifier is to nullify the damping effects of the glass. The amplitude additions then build rapidly as sound wave energy gets stored in the glass, seen as rim vibrations, and the glass breaks.

Glass at 5mm Rim Excursion, before Breakage

Figure 2. Still image from a wine glass experiment video.

The source for the still image is the wine glass experiment Quicktime video found at Blaze Labs,

`http://www.blazelabs.com/f-p-glass.asp`

Soldiers Breaking Cadence

The collapse of the Broughton bridge near Manchester, England in 1831 is blamed for the now–standard practise of breaking cadence when soldiers cross a bridge.

Bridges like the Broughton bridge have many natural low frequencies of vibration, so it is possible for a column of soldiers to vibrate the bridge at one of the bridge's natural frequencies. The bridge locks onto the frequency while the soldiers continue to add to the excursions with every step, causing larger and larger bridge oscillations.

Millenium Bridge over The Thames

In 2000, the Millenium Bridge was opened in London, and closed shortly thereafter, due to unusual vibrations caused by the foot traffic. It was re-opened in 2002 after spending an additional 5 million pounds on bridge re-design. Read about the bridge here:

http://en.wikipedia.org/wiki/Millennium_Bridge_(London).

Figure 3. London's Millenium Suspension Footbridge over the Thames.

Definition of Practical Resonance

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching characterizes the event. This ideal situation never happens in the physical world, because *damping is always present*. In the presence of damping $c > 0$, it will be established below that *only bounded solutions exist* for the forced spring-mass system

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \tag{2}$$

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible solution (a more precise definition appears below). It will be shown that this happens for the condition

$$\omega = \sqrt{k/m - c^2/(2m^2)}, \quad k/m - c^2/(2m^2) > 0. \tag{3}$$

Pure resonance $\omega = \omega_0 \equiv \sqrt{k/m}$ is the limiting case obtained by setting the damping constant c to zero in condition (3). This strange but predictable interaction exists between the damping constant c and the size of solutions, relative to the external frequency ω, even though all solutions remain bounded.

Boundedness of the Homogeneous Solution

The decomposition of $x(t)$ into homogeneous solution $x_h(t)$ and particular solution $x_p(t)$ gives some intuition into the complex relationship between the input frequency ω and the size of the solution $x(t)$.

The homogeneous solution. For positive damping, $c > 0$, equation (2) has homogeneous solution $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$ where according to the *recipe* the basis elements x_1 and x_2 are given in terms of the roots of the characteristic equation $mr^2 + cr + k = 0$, as classified by the discriminant $D = c^2 - 4mk$, as follows:

Case 1, $D > 0$	$x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$ with r_1 and r_2 negative.
Case 2, $D = 0$	$x_1 = e^{r_1 t}$, $x_2 = te^{r_1 t}$ with r_1 negative.
Case 3, $D < 0$	$x_1 = e^{\alpha t}\cos\beta t$, $x_2 = e^{\alpha t}\sin\beta t$.

Symbols α, β satisfy $\beta > 0$ and $\alpha = -c/(2m) < 0$.

It follows that $x_h(t)$ contains a negative exponential factor, regardless of the positive values of m, c, k. Then $y_h(t)$ is *bounded*.

Transient Solution

A solution $x(t)$ is called a **transient solution** provided it satisfies the relation $\lim_{t\to\infty} x(t) = 0$. The conclusion:

> The homogeneous solution $x_h(t)$ of the equation $mx''(t) + cx'(t) + kx(t) = 0$ is a transient solution for all positive values of m, c, k.

A transient solution graph $x(t)$ for large t lies atop the axis $x = 0$, as in Figure 4, because $\lim_{t\to\infty} x(t) = 0$.

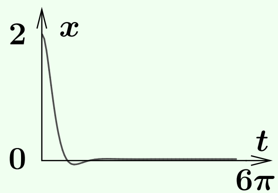

Figure 4. Transient oscillatory solution $x = 2e^{-t}(\cos t + \sin t)$ of the differential equation $x'' + 2x' + 2x = 0$.

Undetermined Coefficients

The method of undetermined coefficients applied to

$$mx'' + cx' + kx = F_0 \cos \omega t$$

gives a trial solution of the form

$$x(t) = A \cos \omega t + B \sin \omega t$$

with coefficients A, B satisfying the equations

$$\begin{aligned} (k - m\omega^2)A &+ (c\omega)B &= F_0, \\ (-c\omega)A &+ (k - m\omega^2)B &= 0. \end{aligned} \tag{4}$$

Solving (4) with Cramer's rule or elimination produces the solution

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}. \tag{5}$$

Steady-State Solution and x_p

The **steady–state** solution, periodic of period $2\pi/\omega$, is given by

$$
\begin{aligned}
x_p(t) &= \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2}\left((k - m\omega^2)\cos\omega t + (c\omega)\sin\omega t\right) \\
&= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\cos(\omega t - \alpha),
\end{aligned} \tag{6}
$$

where α is defined by the phase–amplitude relations

$$
\begin{aligned}
&C\cos\alpha = k - m\omega^2, \quad C\sin\alpha = c\omega, \\
&C = F_0/\sqrt{(k - m\omega^2)^2 + (c\omega)^2}.
\end{aligned} \tag{7}
$$

The terminology **steady–state** refers to that part $x_{SS}(t)$ of the solution $x(t)$ that remains when the transient portion is removed, that is, when all terms containing negative exponentials are removed. As a result, for large T, the graphs of $x(t)$ and $x_{SS}(t)$ on $t \geq T$ are the same. This feature of $x_{SS}(t)$ allows us to find its graph directly from the graph of $x(t)$. We say that $x_{SS}(t)$ is **observable**, because it is the solution visible in the graph after the **transients** (negative exponential terms) die out.

Practical Resonance

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible steady–state amplitude. Mathematically, this happens exactly when the amplitude function $C = C(\omega)$ defined in (7) has a maximum. If a maximum exists on $0 < \omega < \infty$, then $C'(\omega) = 0$ at the maximum. The power rule implies

$$
\begin{aligned}
C'(\omega) &= \frac{-F_0}{2}\,\frac{2(k-m\omega^2)(-2m\omega)+2c^2\omega}{((k-m\omega^2)^2+(c\omega)^2)^{3/2}} \\
&= \omega\left(2mk - c^2 - 2m^2\omega^2\right)\frac{C(\omega)^3}{F_0^2}
\end{aligned}
\tag{8}
$$

If $2km - c^2 \le 0$, then $C'(\omega)$ does not vanish for $0 < \omega < \infty$ and hence there is no maximum. If $2km - c^2 > 0$, then $2km - c^2 - 2m^2\omega^2 = 0$ has exactly one root $\omega = \sqrt{k/m - c^2/(2m^2)}$ in $0 < \omega < \infty$ and by $C(\infty) = 0$ it follows that $C(\omega)$ is a maximum.

Practical resonance for $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ occurs precisely when the external frequency ω is tuned to $\omega = \sqrt{k/m - c^2/(2m^2)}$ and $k/m - c^2/(2m^2) > 0$.

Visualization of Practical Resonance

In Figure 5, the amplitude of the steady–state periodic solution is graphed against the external natural frequency ω, for the differential equation $x'' + cx' + 26x = 10\cos\omega t$ and damping constants $c = 1, 2, 3$. The practical resonance condition is $\omega = \sqrt{26 - c^2/2}$. As c increases from 1 to 3, the maximum point $(\omega, C(\omega))$ satisfies a monotonicity condition: both ω and $C(\omega)$ decrease as c increases. The maxima for the three curves in the figure occur at $\omega = \sqrt{25.5}, \sqrt{24}, \sqrt{21.5}$. Pure resonance occurs when $c = 0$ and $\omega = \sqrt{26}$.

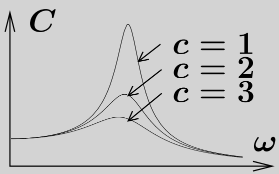

Figure 5. Practical resonance for $x'' + cx' + 26x = 10\cos\omega t$: amplitude $C = 10/\sqrt{(26-\omega^2)^2 + (c\omega)^2}$ versus external frequency ω for $c = 1, 2, 3$.