Introduction to Linear Algebra 2270-3

Final Project Fall 2008

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Instructions. Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Please solve enough parts to make 100% on each chapter.

Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100 for each chapter. Extra credits from one chapter do not apply to other chapters.

Final drafts are expected. Answer checks are expected when applicable. Please submit only complete presentations.

Submit one stapled package of problems, organized with chapter header sheets, in the order they appear below.

Keep this page for your records.

Name ________________________ **2270-3 Final Project**

Ch3. (Subspaces of $\mathcal{R}^n$ and Their Dimensions)

☐ [20%] Ch3(a): Let $A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 & 4 \end{bmatrix}$. Find bases for the image and kernel of A.

☐ [30%] Ch3(b): Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be the columns of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Define $T(\mathbf{x}) = A\mathbf{x}$. Find the matrix of T relative to the basis $\mathbf{v}_1 + 3\mathbf{v}_3$, $\mathbf{v}_2 + 4\mathbf{v}_3$, $\mathbf{v}_1 + 5\mathbf{v}_4$, $\mathbf{v}_1 + \mathbf{v}_2$.

☐ [30%] Ch3(c): Let V be the vector space of all twice continuously differentiable functions $f(x)$ defined on $0 \le x \le 1$. Let S be the subset of V defined by $f(1) = f'(0) + \int_0^1 f''(x)x^3dx$, $f'(1/3) = f(1/3)$. Prove that S is a subspace of V.

☐ [10%] Ch3(d): Prove that the kernel of an $m \times n$ matrix defines a subspace S of $\mathcal{R}^n$.

☐ [10%] Ch3(e): Find a basis for the subspace $S = \mathbf{span}\{e^x, \sin x, 1 - \sin x, 2 + x, 1 + x\}$, in the linear space V of all functions on the real line.

☐ [10%] Ch3(f): Prove that the intersection S of three subspaces S_1, S_2 and S_s of a linear space V is also a subspace of V.

☐ [20%] Ch3(g): Let V be the vector space of all data packages $\mathbf{v} = \begin{pmatrix} f \\ x_0 \\ y_0 \end{pmatrix}$, where f is a continuous function defined on $0 \le x \le 1$ and x_0, y_0 are real values. Define $\boxed{+}$ and $\boxed{\cdot}$ componentwise. Let S be the subset of V defined by $f(0) = f(1)$, $f(1/2) + y_0 = 0$. Prove or disprove: S is a subspace of V.

Please start your solutions on this page. Staple on additional pages.

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Ch4. (Linear Spaces)

☐ [20%] Ch4(a): Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let V be the linear space of all 3×3 matrices. Let S be the set of all 3×3 matrices A such that $\mathbf{x}$ belongs to the image of A. Prove or disprove: S is a subspace of V.

☐ [20%] Ch4(b): Let V be the linear space of all functions $f(x) = c_0 + c_1 x + c_2 x^2$. Define $T(f) = c_1(1 + x) + c_2(1 - x)^2$ from V to V. Find bases for the image and kernel of T and report the rank and nullity of T.

☐ [30%] Ch4(c): Let V be the linear space of all real 5×5 matrices M. Let T be defined on V by $T(\mathbf{M}) = \mathbf{N}$ where $\mathbf{N} = \mathbf{M}$ except for the upper triangle, which is all zeros, and the last two diagonal elements, which are zeros. Find bases for the image and kernel of T.

☐ [30%] Ch4(d): Let $A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$. Find the set W of all matrices B not similar to A. For example, W contains the matrix $B = 0$, because $AS = SB$ implies $AS = 0$ and then $A = 0$, a false statement, meaning $B = 0$ is not similar to A.

Please start your solutions on this page. Staple on additional pages.

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Ch5. (Orthogonality and Least Squares)

☐ [30%] Ch5(a): Find the orthogonal projection of $\mathbf{v}$ onto $V = \mathbf{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, given

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

☐ [20%] Ch5(b): Derive the equations for m and b in the least squares fit of $y = mx + b$ to data points (x_i, y_i), $i = 1, \ldots, n$. State what you assume and try to prove the result from the normal equations in the theory of least squares.

☐ [20%] Ch5(c): Let A be 4×3 with kernel zero. Prove or give a counterexample: $\dim\left(\mathbf{im}(A^T A)\right) + \dim\left(\mathbf{ker}\,(A)\right) = 3$.

☐ [30%] Ch5(d): Consider the linear space V of polynomials $f(t) = c_0 + c_1 t + c_2 t^2$ on $0 \le t \le 1$ with inner product

$$< f, g >= \int_0^1 f(t) g(t) dt.$$

Find a basis for the subspace S of all f in V orthogonal to $1 + t^2$ satisfying the additional restriction equation $f(1/4) = 0$.

☐ [20%] Ch5(e): Find the Gram-Schmidt orthonormal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ for the following independent set:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

☐ [20%] Ch5(f): Find the QR-factorization of $A = \begin{pmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$.

☐ [10%] Ch5(g): Derive the normal equation in the theory of least squares.

☐ [10%] Ch5(h): State and prove the Near Point Theorem.

Please start your solutions on this page. Staple on additional pages.

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Ch6. (Determinants)

☐ [25%] Ch6(a): Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(4B^{-1}(C^T)^{-3})$, where B^{-1} is the inverse of B and C^T is the transpose of C. Part of the problem is to prove that B is invertible.

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & ? & 0 & 0 \\ ? & ? & ? & -3 \end{pmatrix}, \quad C = \begin{pmatrix} ? & 3 & ? & 0 \\ -6 & -3 & ? & 0 \\ -3 & 6 & ? & ? \\ 2 & 1 & ? & -5 \end{pmatrix}$$

☐ [25%] Ch6(b): Assume $A = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is 3×3 and $B = \mathbf{aug}(\mathbf{v}_1 + 2\mathbf{v}_3, 2\mathbf{v}_3 - \mathbf{v}_2, 3\mathbf{v}_2 - 2\mathbf{v}_3)$. Suppose $\det(A + B) + 3\det(A^2) = 0$. Find all possible values of $\det(A)$.

☐ [25%] Ch6(c): Prove from the Four Rules that $\det(A) = 0$ if $\mathbf{rref}(A)$ has a row of zeros.

☐ [25%] Ch6(d): Assume given 3×3 matrices A, B. Suppose $E_5E_4E_3B = E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a combination, a multiply by -2, a combination, a swap and a multiply by 2. Assume $\det(A) = -1$. Find $\det(3A^3B^2)$.

☐ [25%] Ch6(e): Evaluate $\det(A)$ by any hybrid method, where symbol x is a variable. Then solve $\det(A) = 0$ for x (two answers).

$$A = \begin{pmatrix} -x & 1 & 3 & 0 & 0 \\ -1 & 1 + x/2 & 0 & 0 & 1 \\ -1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 3 & -3 & 0 \\ 1 & 2 & 5 & 4 & 1 \end{pmatrix}$$

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Ch7. (Eigenvalues and Eigenvectors)

☐ [20%] Ch7(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 1 & 2 & 3 & 4 & -1 \end{pmatrix}$. To save time, **do not** find eigenvectors!

☐ [20%] Ch7(b): Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, find an invertible matrix P and a diagonal matrix D such that $AP = PD$.

☐ [20%] Ch7(c): Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

You are given that two of the eigenpairs are

$$\left(1, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display explicitly Fourier's model for A.

☐ [20%] Ch7(d): Consider a discrete dynamical system $\mathbf{x}(n+1) = A\mathbf{x}(n)$. Given A and $\mathbf{x}(0)$ below, find exact formulas for the vectors $\mathbf{x}(n)$ and $\lim_{n\to\infty} \mathbf{x}(n)$.

$$A = \frac{1}{7}\begin{pmatrix} 5 & 1 \\ -2 & 8 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 72 \\ 90 \end{pmatrix}.$$

Please start your solutions on this page. Staple on additional pages.