Mathematics 5410
Phase Diagrams

Example. Plot the phase diagram of the linear dynamical system

$$x' = -2x + 3y, \quad y' = -3x - 2y.$$

Solution: The `Maple V.4` code which does the plot appears below.

```
restart:with(DEtools):
h:=0.2:k:=0.2:
de:=[diff(x(t),t)=-2*x(t)+3*y(t),
     diff(y(t),t)=-3*x(t)-2*y(t)]:
vars:=[x(t),y(t)]:
inits:=[seq(seq([0,i*h,j*k],i=-2..2),j=-2..2)]:
dsolve(de,vars);
phaseportrait(de,vars,t=0..2*Pi,inits);
```

Problem 1. Plot a phase diagram for the matrix systems $X' = AX$, given matrix A below, and classify as *center*, *stable spiral*, *unstable spiral*, *proper node*, *improper node*, *saddle*. Subdivide the proper nodes into the two cases *deficient node* and *star node*, as in Borrelli–Coleman, page 389.

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

$$A_{10} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}.$$

Problem 2. Reproduce the soft spring phase diagram for (see Borrelli–Coleman page 153)

$$x' = y, \quad y' = -10x + 0.2x^3 - 0.2y - 9.8.$$

Classify the equilibria (see Borrelli–Coleman page 389)

$$(-1, 0), \quad ((1 + \sqrt{197})/2, 0), \quad ((1 - \sqrt{197})/2, 0).$$

Hint: Use after the `inits` entry, the options

```
x=-12.5..12.5, y=-25..25,
arrows=none, stepsize=0.05
```

to make a finer plot without the direction field. Read the `maple` help on `phaseportrait`. The object is to redefine `inits` in the example above. There are about 12 significant initial conditions to plot, based upon Figure 3.1.2, page 153. Choose the three equilibria for a start, then add others until the figure matches the one in the text.

Problem 3. Consider the soft spring phase diagram for (see Borrelli–Coleman page 153)

$$x' = y, \quad y' = -10x + 0.2x^3 - 0.2y - 9.8.$$

Make three individual phase diagrams for each of the three linearized equations about the equilibrium points

$$(-1, 0), \quad ((1 + \sqrt{197})/2, 0), \quad ((1 - \sqrt{197})/2, 0).$$

The linearized differential equations about these equilibria are (see Borrelli–Coleman, page 154, for the first one):

$$x' = y, \quad y' = -9.4(x + 1) - 0.2y,$$

$$x' = y, \quad y' = (197 + 3\sqrt{197})(x - 1/2 - \sqrt{197}/2)/10 - 0.2y,$$
$$x' = y, \quad y' = (197 - 3\sqrt{197})(x - 1/2 + \sqrt{197}/2)/10 - 0.2y.$$

Problem 4. Justify mathematically the linearized differential equations given in Problem 3. This work is to be handwritten and full of detail.

Taylor's Theorem. Let $F(X)$ be a function from R^n into R^n, twice continuously differentiable. Let X_0 be a given point. Then

$$F(X) = F(x_0) + J(X - X_0) + \mathcal{R}$$

where J is the Jacobian matrix of F at $X = X_0$ and the remainder $\mathcal{R}$ satisfies $|\mathcal{R}| \le K|X - X_0|^2$ as X approaches X_0.

The Jacobian matrix has entries $\partial F_i/\partial x_j$, that is, the columns of J are the vector partials $\partial_{x_j} F(X_0)$.

Linearized Equation. At an equilibrium point X_0, the dynamical system $X' = F(X)$ has linearization $X' = J(X - X_0)$. This equation is formally obtained from its nonlinear counterpart $X' = F(X)$ by dropping the Taylor remainder and observing that, by assumption, $F(X_0) = 0$.

Example. Compute the Jacobian matrix at $X_0 = (-1, 0)$ for the vector function

$$F(x, y) = \begin{pmatrix} y \\ -10x + 0.2x^3 - 0.2y - 9.8 \end{pmatrix}$$

and find the linearized system for $X' = F(X)$.

Solution: The partials are

$$\begin{aligned}
\partial_x F &= \partial_x \begin{pmatrix} y \\ -10x + 0.2x^3 - 0.2y - 9.8 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -10 + 0.6x^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -10 + 0.6(-1)^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -9.4 \end{pmatrix}, \\
\partial_y F &= \partial_y \begin{pmatrix} y \\ -10x + 0.2x^3 - 0.2y - 9.8 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -0.2 \end{pmatrix}, \\
J &= \begin{pmatrix} 0 & 1 \\ -9.4 & -0.2 \end{pmatrix}.
\end{aligned}$$

The linearized system is $X' = J(X - X_0)$, or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -9.4 & -0.2 \end{pmatrix} \begin{pmatrix} x + 1 \\ y \end{pmatrix}.$$