Differential Equations and Linear Algebra 2250-2

10:45 Midterm Exam 3, Spring 2006

Version 3

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuously differentiable functions defined on $-2 \le x \le 2$. Define V to be the set of all functions $f(x)$ in S such that $\int_0^2 x f'(x)dx = 0$. Prove that V is a subspace of S, by using the Subspace Criterion.

(b) [30%] Let S be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let V be the subset of S defined by the dot product equations $(\mathbf{a}+\mathbf{b}) \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{b} \cdot \mathbf{x} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in S. Prove that V is a subspace of S.

(c) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details. Briefly, show the entire snapshot sequence to the **rref**, then display the general solution with variables t_1, t_2, ...

$$\begin{array}{ccccccccc} a & + & b & - & 2c & + & d & = & 2 \\ & + & b & + & 2c & + & & = & 0 \\ a & + & 2b & + & & + & d & = & 2 \\ a & + & 3b & + & 2c & + & d & = & 2 \end{array}$$

Solution 1(a). Use the subspace criterion: (a) Given f and g in V, write details to show $f+g$ is in V; (b) Given f in V and k constant, write details to show kf is in V. Let $h(x) = x$, which is a function in S. Details for (a): Given $\int_0^2 f'(x)h(x)dx = 0$ and $\int_0^2 g'(x)h(x)dx = 0$, add the equations to obtain the equation $\int_0^1 (f'(x) + g'(x))h(x)dx = 0$. This finishes (a). Details for (b): Given $\int_0^2 f'(x)h(x)dx = 0$ and k constant, multiply the equation by k and re-arrange factors to obtain the new equation $\int_0^2 (kf'(x))h(x)dx = 0$. This proves (b).

Solution 1(b). Let $\mathbf{a}$ and $\mathbf{b}$ be given. Let A be the matrix whose rows are $\mathbf{a}+\mathbf{b}$, $\mathbf{b}$, $\mathbf{0}$. Then the restriction equations given are equivalent to $A\mathbf{x} = \mathbf{0}$. By Theorem 2 in Edwards-Penney, V is a subspace of S.

Solution 1(c). The answer is $\begin{pmatrix} 2+4t_1-t_2 \\ -2t_1 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

2. **(ch5)** Complete (a) and either (b) or (c). Do not do both (b) and (c).

(a) [30%] Given $4x''(t) + 4x'(t) + 5x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 5$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [70%] Display by variation of parameters a particular solution x_p for the equation $x'' + 2x' = f(t)$. Leave the answer in unevaluated integral form. Evaluate all symbols except $f(t)$ appearing in (33) of Edwards-Penney.

(c) [70%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 2x = 5\sin(t)$.

Solution 2(a).
Use $4r^2 + 4r + 5 = 0$ and the quadratic formula to obtain roots $r_1 = -1/2 + i$, $r_2 = -1/2 - i$. Case 3 of the recipe gives $x(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$. This is under-damped.

Solution 2(b).
Solve $x'' + 2x' = 0$ by the recipe to get $x_h = c_1x_1 + c_2x_2$, $x_1 = 1$, $x_2 = e^{-2t}$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = -2e^{-2t}$. Then

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt$$

becomes

$$x_p = \int \frac{f}{2} dt + e^{-2t} \int \frac{-f(t)e^{2t}}{2} dt.$$

Solution 2(c). The trial solution is $x = d_1 \cos t + d_2 \sin t$. Substitute the trial solution to obtain the answers $d_1 = -2$, $d_2 = 1$. The unique periodic solution x_{SS} is extracted from the general solution $x = x_h + x_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). If $x = x_h + x_p$ and $x_p = d_1 \cos t + d_2 \sin t = -2\cos t + \sin t$, then all terms of $x_h = c_1e^{-t}\cos t + c_2e^{-t}\sin t$ are crossed out, giving the steady-state solution

$$x_{SS} = -2\cos t + \sin t.$$

3. **(ch5)** Complete all parts below.

(a) [75%] Determine for $y^v - 9y''' = 2xe^{3x} + 3x^3 + 2\cos 3x + \sin 3x$ the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps **1** and **2**, but skip steps **3** and **4**)! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r^2 - 4)^3(r + 2)(r^2 + 6r + 10)^2 = 0$.

Solution 3(a).
The homogeneous solution is $y_h = c_1 + c_2x + c_3x^2 + c_4e^{2x} + c_5e^{-2x}$, because the characteristic polynomial has roots 0, 0, 0, 2, −2.

1 An initial trial solution y is constructed for atoms $1, x, x^2, x^3, e^{3x}, xe^{3x}, \cos 3x, \sin 3x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= d_1 + d_2x + d_3x^2 + d_4x^3, \\ y_2 &= (d_5 + d_6x)e^{3x}, \\ y_3 &= d_7\cos 3x + d_8 \sin 3x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

2 The fixup rule $[d_j \to d_jx^{s_j}]$ is applied individually to each atom to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= x^3(d_1 + d_2x + d_3x^2 + d_4x^3), \\ y_2 &= x(d_5 + d_6x)e^{3x}, \\ y_3 &= d_7\cos 3x + d_8 \sin 3x. \end{aligned}$$

The powers x^{s_j} multiplied in each case are designed to eliminate terms in the initial trial solution which duplicate atoms appearing in the homogeneous solution y_h. The factor is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. By design, unrelated atoms are unaffected by the fixup rule [$s_j = 0$ in this case and factor $x^0 = 1$] and that is why y_3 was unaltered.

3 Undetermined coefficient step skipped, according to the problem statement.

4 Undetermined coefficient step skipped, according to the problem statement.

Solution 3(c).
Write $(r^2-4)^3(r+2)(r^2+6r+10)^2 = 0$ as $(r-2)^3(r+2)^4(r^2+6r+10)^2 = 0$, then $y = u_1e^{2x} + u_2e^{-2x} + u_3e^{-3x}\cos x + u_4e^{-3x}\sin x$. The polynomials are $u_1 = c_1 + c_2x + c_3x^2$ (3 terms for multiplicity 3), $u_2 = c_4 + c_5x + c_6x^2 + c_7x^3$ [4 terms for multiplicity 4], $u_3 = c_8 + c_9x$, $u4 = c_{10} + c_{11}x$.

4. **(ch6)** Complete all of the items below.

(a) [40%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 5 & 1 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [60%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Find a column of P so that 6 appears in the same column of D.

Solution 4(a).
Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ to obtain the characteristic polynomial $(4-\lambda)(5-\lambda)[(1-\lambda)(1-\lambda)+20] = 0$. The eigenvalues are the roots: $\lambda = 4, 5, 1+2\sqrt{5}i, 1-2\sqrt{5}i$. Used here was the *cofactor rule* for determinants. Sarrus' rule does not apply for 4×4 determinants (an error) and the triangular rule likewise does not directly apply (another error).

Solution 4(b).
Details: According to the theory of diagonalizable matrices, P is the matrix package of eigenvectors and D is the matrix package of eigenvalues. There are $3! = 6$ possible orderings to make these packages, hence 6 possible choices exist, all of which are correct. In all cases, an eigenpair is entered in the same location in each package. One way to package:

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The problem is solved without displaying the packages. It suffices to recognize that $A\mathbf{x} = 6\mathbf{x}$ defines the eigenvector $\mathbf{x}$ corresponding to $\lambda = 6$. Solving gives column 1 of P above.

5. **(ch6)** Complete all parts below.

Consider a given 3×3 matrix A having three eigenpairs

$$5, \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}; \quad -3, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

(a) [50%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.

(c) [30%] Let B be a certain 2×2 matrix. Fourier's model for the computation of $B\mathbf{x}$ is known to be

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{implies}$$
$$B\mathbf{x} = -c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find B.

Solution 5(a).
Answer: The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Solution 5(b).

$$AP = PD \text{ implies } A = PDP^{-1}.$$

Solution 5(c).
Answer: Fourier's model implies the eigenpairs of B are given by

$$\left(-1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-3, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

Then $D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ implies by $BP = PD$ that $B = PDP^{-1}$ and finally

$$B = \begin{pmatrix} -2 & 1/2 \\ 2 & -2 \end{pmatrix}.$$