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1. (rref)
Determine a, b such that (1) the system has no solution and (2) the system has a unique solution.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ 5x & + & 10y & + & 2z & = & 2 \\ 6x & + & ay & + & bz & = & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 5 & 10 & 2 & 2 \\ 6 & a & b & 2 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & a-12 & b-6 & -4 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & a-12 & b-6 & -4 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & a-12 & b-6 & -4 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & a-12 & 0 & -4+6-b \end{array}\right)$$

(1) No solution: $a-12=0$, $b\neq 2$
signal eq "$0=1$"

(2) Unique solution: $a-12\neq 0$.
Because of Three lead variables

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2. (vector spaces)

(a) [25%] Let V be the vector space of functions $f(t) = c_1 + c_2e^{-t} + c_3te^t + c_4t^2e^t$, for all values of c_1, c_2, c_3, c_4. Report a basis for V.

(b) [25%] Prove that the set S of all vectors $\mathbf{v}$ in $\mathcal{R}^3$ with $v_2 = 0$ is a subspace.

(c) [50%] Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + y - 3z &= 0, \\ x + 3y - 2z &= 0, \\ 2y + z &= 0, \end{aligned}$$

(a) $1, e^{-t}, te^t, t^2e^t$ = basis

(b) First, $\begin{pmatrix}0\\0\\0\end{pmatrix}$ has component $v_2=0$, so $\vec{0} \in S$.
Second, given $\begin{pmatrix}x_1\\0\\z_1\end{pmatrix}, \begin{pmatrix}x_2\\0\\z_2\end{pmatrix}$ in S, then

$$c_1\begin{pmatrix}x_1\\0\\z_1\end{pmatrix} + c_2\begin{pmatrix}x_2\\0\\z_2\end{pmatrix} = \begin{pmatrix}c_1x_1+c_2x_2\\0\\c_1z_1+c_2z_2\end{pmatrix} \text{ is in } S.$$

Proof is complete.

(c)
$$\begin{pmatrix}1&1&-3\\1&3&-2\\0&2&1\end{pmatrix} \cong \begin{pmatrix}1&1&-3\\0&2&1\\0&2&1\end{pmatrix} \cong \begin{pmatrix}1&1&-3\\0&1&1/2\\0&0&0\end{pmatrix}$$

$$\cong \begin{pmatrix}1&0&-3.5\\0&1&0.5\\0&0&0\end{pmatrix}$$

$$x = 3.5t, \quad y = -0.5t, \quad z = t,$$

Basis = $\begin{pmatrix}7/2\\-1/2\\1\end{pmatrix}$ or $\begin{pmatrix}7\\-1\\2\end{pmatrix}$

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3. (independence)

Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. State and apply a test that shows **u**, **v**, **w** are independent.

Test: $\det(\text{aug}(u,v,w)) \neq 0 \iff u,v,w$ are independent

$$\begin{vmatrix} 1 & 4 & 1 \\ -1 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 0 + (1)\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix}$$

$$= 14 + 0 + 5$$
$$= 19$$
$$\neq 0$$

u, v, w are independent.

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4. **(determinants and elementary matrices)**
Let A and B be two 7×7 matrices such that two combination rules applied to BA give the original matrix A. Explain precisely why either $\det(A) = 0$ or $\det(B) = 1$.

$A = E_1 E_2 BA$

Matrices E_1, E_2 are combo rules.
$\det(E_1) = \det(E_2) = 1$

$\det(A) = \det(E_1)\det(E_2)\det(B)\det(A)$ prod rule for determinants

$\det(A) = 1 \cdot 1 \cdot \det(B)\det(A)$

$\det(A)(1 - \det(B)) = 0$

$\therefore \det(A) = 0$ or $1 - \det(B) = 0$

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5. (inverses and Cramer's rule)

Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 2x & 1 & 1 \\ 0 & x & 1 & 0 \end{pmatrix}$.

A^{-1} exists $\iff \det(A) \neq 0$.

cofactor exp col 4

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 2x & 1 & 1 \\ 0 & x & 1 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & x & 1 \end{vmatrix}$$

$$= -\left(1 \begin{vmatrix} 0 & -3 \\ x & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} \right)$$

$$= -(3x - 4)$$

$$\boxed{x \neq \frac{4}{3} \iff A^{-1} \text{ exists}}$$

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