Independent Particles in a Dynamical Random Environment

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Abstract We study the motion of independent particles in a dynamical random environment on the integer lattice. The environment has a product distribution. For the multidimensional case, we characterize the class of spatially ergodic invariant measures. These invariant distributions are mixtures of inhomogeneous Poisson product measures that depend on the past of the environment. We also investigate the correlations in this measure. For dimensions one and two, we prove convergence to equilibrium from spatially ergodic initial distributions. In the one-dimensional situation we study fluctuations of the net current seen by an observer traveling at a deterministic speed. When this current is centered by its quenched mean its limit distributions are the same as for classical independent particles.

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1 Introduction and Results

This paper studies particles that move on the integer lattice $\mathbb{Z}^d$. Particles interact through a common environment that specifies their transition probabilities in space and time. The environment is picked randomly at the outset and fixed for all time. Given the environment, particles evolve independently, governed by the transition probabilities specified by the environment.

We have two types of results. First we characterize those invariant distributions for the particle process that satisfy a spatial translation invariance. These turn out to be mixtures of inhomogeneous Poisson product measures that depend on the past of the environment. Poisson is expected, in view of the classical result that a system of independent random walks has a homogeneous Poisson invariant distribution [5, Sect. 8.5]. For $d = 1, 2$, we use coupling ideas from [7] (as presented in [18]) to prove convergence to this equilibrium from spatially invariant initial distributions.

In the one-dimensional case we study fluctuations of the particle current seen by an observer moving at the characteristic speed. In the present setting the characteristic speed is simply the mean speed v of the particles. More generally, the characteristic speed is the derivative $H'(\rho)$ of the flux H as a function of particle density ρ. The flux $H(\rho)$ is the mean rate of flow across a fixed bond of the lattice when the system is stationary with density ρ. For independent particles $H(\rho) = v\rho$.

It is expected, and supported by known rigorous results, that the current fluctuations are of order $n^{1/4}$ with Gaussian limits if the macroscopic flux H is linear, and of order $n^{1/3}$ with Tracy-Widom type limits if the flux H is strictly convex or concave. In statistical physics terminology, the former is the Edwards-Wilkinson (EW) universality class, and the latter the Kardar-Parisi-Zhang (KPZ) universality class. (See [3] for the physics perspective on these matters, and [4, 20] for mathematical reviews). Our motivation is to investigate the effect of a random environment in the EW class. We find that, when the current is centered by its quenched mean and the environment is averaged out, the fluctuation picture in the dynamical environment is the same as that for classical independent random walks [10, 19]. Consistent with EW universality, the current fluctuations have magnitude $t^{1/4}$ and occur on a spatial scale of $t^{1/2}$ where t denotes the macroscopic time variable.

There is an interesting contrast with the case of static environment investigated in [15]. In the static environment, the quenched mean of the current has fluctuations of magnitude $t^{1/2}$ and converges weakly to a Brownian motion. Our results suggest that under a dynamic environment the quenched mean of the current has fluctuations of magnitude $t^{1/4}$ and that when the particle system is stationary in time these fluctuations are governed by a fractional Brownian motion with Hurst parameter $1/4$.

Other work on the motion of independent particles in a random environment includes articles [9, 14].

We turn to a description of the process and then the results.

1.1 The Particle Process and Its Invariant Distributions

The particles follow independent random walks in a common dynamical random environment (RWRE). More precisely, they move in a space-time environment $\omega = (\omega_{x,s})_{(x,s)\in\mathbb{Z}^d\times\mathbb{Z}}$ indexed by a discrete time variable s and a discrete space variable x. The environment at space-time point $(x,s)\in\mathbb{Z}^d\times\mathbb{Z}$ is a vector $\omega_{x,s} = (\omega_{x,s}(z) : z\in\mathbb{Z}^d,\ |z|\leqslant R)$ of jump probabilities that satisfy

$$0\leqslant \omega_{x,s}(z)\leqslant 1 \quad\text{and}\quad \sum_{z\in\mathbb{Z}^d\,:\,|z|\leqslant R}\omega_{x,s}(z) = 1. \tag{1.1}$$

R is a fixed finite constant that specifies the range of jumps. From a space-time point (x,s) admissible jumps are to points $(y,s+1)$ such that $|y-x|\leqslant R$. In environment ω the transition probabilities governing the motion of a $\mathbb{Z}^d$-valued walk $X_{\bullet} = (X_s)_{s\in\mathbb{Z}_+}$ are

$$P^\omega[X_{s+1}=y\,|\,X_s=x] = \pi^\omega_{s,s+1}(x,y)\equiv\omega_{x,s}(y-x). \tag{1.2}$$

P^ω is the quenched probability measure on the path space of the walk $X_\bullet$. The environment is "dynamical" because at each time s the particle sees a new environment $\bar\omega_s = (\omega_{x,s} : x\in\mathbb{Z}^d)$.

$(\Omega,\mathfrak{S})$ denotes the space of environments ω satisfying the above assumptions, endowed with the product topology and its Borel σ-algebra $\mathfrak{S}$. The environment restricted to levels $s\in\{m,\dots,n\}$ is denoted by

$$\bar\omega_{m,n} = (\bar\omega_s)_{m\leqslant s\leqslant n} = (\omega_{x,s} : m\leqslant s\leqslant n, x\in\mathbb{Z}^d).$$

Environments at levels generate σ-algebras $\mathfrak{S}_{m,n} = \sigma\{\bar\omega_{m,n}\}$. In these formulations $m=-\infty$ or $n=\infty$ are also possible. $T_{x,s}$ is the shift on Ω, that is $(T_{x,s}\omega)_{y,t} = \omega_{x+y,s+t}$.

Let $\mathbb{P}$ be a probability measure on Ω such that

$$\text{the probability vectors } (\omega_{x,s})_{(x,s)\in\mathbb{Z}^d\times\mathbb{Z}} \text{ are i.i.d. under } \mathbb{P}. \tag{1.3}$$

We make two nondegeneracy assumptions. The first one guarantees that the quenched walk is not degenerate:

$$\mathbb{P}\{\,\exists\, z\in\mathbb{Z}^d : 0<\omega_{0,0}(z)<1\,\}>0. \tag{1.4}$$

Denote the mean transition kernel by $p(u) = \mathbb{E}\pi_{s,s+1}(x, x+u)$. The second key assumption is that

$$\text{there does not exist } x \in \mathbb{Z}^d \text{ and an additive subgroup } \mathbb{G} \subsetneq \mathbb{Z}^d \text{ such that } \sum_{z\in\mathbb{G}} p(x+z) = 1. \tag{1.5}$$

Another way to state assumption (1.5) is that the averaged walk has span 1, or that it is aperiodic in Spitzer's [22] terminology.

To create a system of particles, let $\{X^{u,j}_{\bullet} : u \in \mathbb{Z}^d,\ j \in \mathbb{N}\}$ denote a collection of random walks on $\mathbb{Z}^d$ such that walk $X^{u,j}_{\bullet}$ starts at site u: $X^{u,j}_0 = u$. When the environment ω is fixed, we use P^ω to denote the joint quenched measure of the walks $\{X^{u,j}_{\bullet}\}$. Under P^ω these walks move independently on $\mathbb{Z}^d$ and each walk obeys transitions (1.2).

Further, assume given an initial configuration $\eta = (\eta(u))_{u\in\mathbb{Z}^d}$ of occupation variables. Variable $\eta(u) \in \mathbb{Z}_+$ specifies the number of particles initially at site u. P^ω_η denotes the quenched distribution of the walks $\{X^{u,j}_{\bullet} : u \in \mathbb{Z}^d,\ 1 \leqslant j \leqslant \eta(u)\}$. Occupation variables for all times $s \in \mathbb{Z}_+$ are then defined by

$$\eta_s(x) = \sum_{u\in\mathbb{Z}^d} \sum_{j=1}^{\eta(u)} \mathbf{1}\{X^{u,j}_s = x\}, \qquad (x, s) \in \mathbb{Z}^d \times \mathbb{Z}_+.$$

When the initial configuration $\eta = \eta_0$ has probability distribution ν we write $P^\omega_\nu(\cdot) = \int P^\omega_\eta(\cdot)\,\nu(d\eta)$ for the quenched distribution of the process.

When the environment is averaged over we drop the superscript ω: for any event A that involves the walks and occupation variables, and any event $B \subseteq \Omega$, $P_\nu(A \times B) = \int_B P^\omega_\nu(A)\,\mathbb{P}(d\omega)$.

It will be convenient to construct initial distributions $\nu = \nu^\omega$ as functions of the environment, so that the quenched process distribution is then $P^\omega_{\nu^\omega}(\cdot) = \int P^\omega_\eta(\cdot)\ \nu^\omega(d\eta)$. But then it will always be the case that ν^ω depends only on the *past* $\bar{\omega}_{-\infty,-1}$ of the environment. Consequently the initial distribution ν^ω and the quenched distribution of the walks $P^\omega(\{X^{x,j}_{\bullet}\} \in \cdot)$ are independent under the product measure $\mathbb{P}$ on the environment. The averaged process distribution is then

$$\int_\Omega P^\omega_{\nu^\omega}(\cdot)\,\mathbb{P}(d\omega) = \int_{\mathbb{Z}_+^{\mathbb{Z}^d}} P_\eta(\cdot)\,\bar{\nu}(d\eta) = P_{\bar{\nu}}(\cdot)$$

where $\bar{\nu}(d\eta) = \int_\Omega \nu^\omega(d\eta)\,\mathbb{P}(d\omega)$ is the averaged initial distribution. In particular, in both quenched and averaged sense, the initial occupation variables $\{\eta_0(x)\}$ are independent of the walks $\{X^{x,j}_{\bullet}\}$.

The first result describes the invariant distributions of the occupation process $\eta_t = (\eta_t(x))_{x\in\mathbb{Z}^d}$. The starting point is an invariant distribution for the environment process seen by a tagged particle: this is the process $T_{X_n,n}\omega$ where $X_{\bullet}$ denotes a walk that starts at the origin. A familiar martingale argument and Green function bounds (Proposition 3.1 in Sect. 3 below) show the existence of an $\mathfrak{S}_{-\infty,-1}$-measurable density function

f on Ω such that $\mathbb{E}(f) = 1$, $\mathbb{E}(f^2) < \infty$, and the probability measure $\mathbb{P}_\infty(d\omega) = f(\omega)\,\mathbb{P}(d\omega)$ is invariant for the Markov chain $T_{X_n,n}\omega$.

For $0 \leqslant \lambda < \infty$ let Γ^λ denote the mean λ Poisson distribution on $\mathbb{Z}_+$. For $0 \leqslant \rho < \infty$ and $\omega \in \Omega$ define the following inhomogeneous Poisson product probability distribution on particle configurations $\eta = (\eta(x))_{x\in\mathbb{Z}^d}$:

$$\mu^{\rho,\omega}(d\eta) = \bigotimes_{x\in\mathbb{Z}^d} \Gamma^{\rho f(T_{x,0}\omega)}\big(d\eta(x)\big). \tag{1.6}$$

(Such a measure is called a Cox process with random intensity $\rho f(T_{x,0}\omega)$). Define the averaged measure by

$$\mu^\rho = \int \mu^{\rho,\omega}\,\mathbb{P}(d\omega). \tag{1.7}$$

Theorem 1.1 *Let the dimension $d \geqslant 1$. Consider independent particles on $\mathbb{Z}^d$ in an i.i.d. space-time environment (as indicated in* (1.3)*), with bounded jumps, under assumptions* (1.4) *and* (1.5)*.*

(a) *For each $0 \leqslant \rho < \infty$, μ^ρ is the unique invariant distribution for the process $\eta_\bullet$ that is also invariant and ergodic under spatial translations and has mean occupation $\int \eta(x)\,d\mu^\rho = \rho$. Furthermore, the tail σ-field of the state space $\mathbb{Z}_+^{\mathbb{Z}^d}$ is trivial under μ^ρ, and under the path measure P_{μ^ρ} the process $\eta_\bullet$ is ergodic under time shifts.*

(b) *Suppose $d = 1$ or $d = 2$. Let ν be a probability distribution on $\mathbb{Z}_+^{\mathbb{Z}^d}$ that is stationary and ergodic under spatial translations and has mean occupation $\rho = \int \eta(x)\,d\nu$. Then if ν is the initial distribution for the process $\eta_\bullet$, the process converges in distribution to the invariant distribution with density ρ: $P_\nu\{\eta_t \in \cdot\} \Rightarrow \mu^\rho$ as $t \to \infty$.*

Part (b) of the theorem is restricted to $d = 1, 2$ because our proof uses recurrence of random walks (see Proposition 2.2).

Two auxiliary Markov transitions q and $\bar{q}$ on $\mathbb{Z}^d$ play important roles throughout much of the paper:

$$\begin{aligned} q(x, y) &= \sum_{z\in\mathbb{Z}^d} \mathbb{E}[\omega_{0,0}(z)\omega_{x,0}(z+y)] \\ &= \begin{cases} \sum_{z\in\mathbb{Z}^d} \mathbb{E}[\omega_{0,0}(z)\omega_{0,0}(z+y)] & x = 0,\, y \in \mathbb{Z}^d \\ \sum_{z\in\mathbb{Z}^d} p(z)p(z+y-x) & x \neq 0,\, y \in \mathbb{Z}^d \end{cases} \end{aligned} \tag{1.8}$$

and

$$\bar{q}(x, y) = \bar{q}(0, y-x) = \sum_{z\in\mathbb{Z}^d} p(z)p(z+y-x). \tag{1.9}$$

Think of q as a symmetric random walk whose transition probability is perturbed at the origin, and of $\bar{q}$ as the corresponding unperturbed homogeneous walk.

For $\theta \in \mathbb{T}^d = (-\pi, \pi]^d$ define characteristic functions

$$\phi^\omega(\theta) = \sum_z \omega_{0,0}(z) e^{i\theta\cdot z}, \tag{1.10}$$

$$\lambda(\theta) = \sum_{z\in\mathbb{Z}^d} q(0,z) e^{i\theta\cdot z} = \mathbb{E}|\phi^\omega(\theta)|^2 \tag{1.11}$$

and

$$\bar\lambda(\theta) = \sum_{z\in\mathbb{Z}^d} \bar q(0,z) e^{i\theta\cdot z} = |\mathbb{E}\phi^\omega(\theta)|^2. \tag{1.12}$$

(We use the bar notation for quantities associated with the homogeneous walk $\bar q$, in addition to a few other particular items such as $\bar\omega_s$ for the environment on level s. In the case of $\bar\lambda(\theta)$ this must not be confused with complex conjugation). Assumption (1.5) implies that the random walk $\bar q$ is not supported on a subgroup smaller than $\mathbb{Z}^d$, hence $\bar\lambda(\theta) < 1$ for $\theta \in \mathbb{T}^d \setminus \{0\}$ [22, p. 67, T7.1]. Define a constant β by

$$\beta = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \frac{1-\lambda(\theta)}{1-\bar\lambda(\theta)}\, d\theta. \tag{1.13}$$

The distribution $q(0,z)$ is not degenerate by assumption (1.4) and hence $\lambda(\theta)$ is not identically 1. Since also $\bar\lambda(\theta) \leqslant \lambda(\theta)$, we see that $\beta \in (0,1]$ is well-defined.

Under the invariant distribution μ^ρ the covariance of the occupation variables is

$$\mathrm{Cov}^{\mu^\rho}[\eta(0), \eta(m)] = \rho^2 \mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = \rho^2 \mathbb{E}[f(\omega) f(T_{m,0}\omega)] - \rho^2, \quad m \in \mathbb{Z}^d. \tag{1.14}$$

The first equality above comes from the structure of μ^ρ: given ω, the occupation variables are independent with means $E^{\mu^{\rho,\omega}}[\eta(m)] = \rho f(T_{m,0}\omega)$. Our next theorem gives a formula for (1.14).

Theorem 1.2 *Let* $d \geqslant 1$. *For* $m \in \mathbb{Z}^d \setminus \{0\}$

$$\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = -\frac{\beta^{-1}}{(2\pi)^d}\int_{\mathbb{T}^d} \cos(\theta\cdot m) \frac{1-\lambda(\theta)}{1-\bar\lambda(\theta)}\, d\theta \tag{1.15}$$

and

$$\mathbb{V}\mathrm{ar}[f(\omega)] = \beta^{-1} - 1. \tag{1.16}$$

The compact analytic formulas (1.13) and (1.15) arise from probabilistic formulas that involve the transitions q and $\bar q$ and the potential kernel of $\bar q$. The probabilistic arguments are somewhat different in the recurrent ($d \leqslant 2$) and transient ($d \geqslant 3$) cases. The reader can find these in Sect. 4.

By the Riemann-Lebesgue lemma we have that

$$\lim_{m\to\infty} \mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = 0.$$

By computing the integral in (1.15) an interesting special case arises:

Corollary 1.3 *For the simplest case where $d = 1$ and $p(x) + p(x+1) = 1$ for some $x \in \mathbb{Z}$, the fixed time occupation variables in the stationary process are uncorrelated:*

$$\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = 0 \quad \textit{for } m \neq 0.$$

1.2 Limit of the Current Process

To study the particle current we restrict to dimension $d = 1$. Define the mean and variance of the averaged walk by

$$v = \sum_{x\in\mathbb{Z}} x p(x) \quad \text{and} \quad \sigma^2 = \sum_{x\in\mathbb{Z}} x^2 p(x) - v^2. \tag{1.17}$$

For $t \in \mathbb{R}_+ = [0, \infty)$ and $r \in \mathbb{R}$, let

$$Y_n(t,r) = \sum_{x>0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X^{x,j}_{\lfloor nt\rfloor} \leqslant \lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor\} - \sum_{x\leqslant 0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X^{x,j}_{\lfloor nt\rfloor} > \lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor\}. \tag{1.18}$$

$Y_n(t,r)$ represents the net right-to-left current of particles seen by a moving observer who starts at the origin and travels to $\lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor$ in time $\lfloor nt\rfloor$.

We look at the current under the following assumptions. Given ω, initial occupation variables obey a product measure that may depend on the past of the environment, but so that shifts are respected. Precisely,

given the environment ω, initial occupation variables $(\eta_0(x))_{x\in\mathbb{Z}}$ have distribution $\mu^\omega(d\eta_0) = \bigotimes_{x\in\mathbb{Z}} \mu^\omega_x(d\eta_0(x))$ where μ^ω_x is allowed to depend measurably on $\bar{\omega}_{-\infty,-1}$. Furthermore, $\mu^\omega_x = \mu_0^{T_{x,0}\omega}$. (1.19)

Let P^ω denote the quenched distribution $P^\omega_{\mu^\omega}$ of initial occupation variables and walks, and $P = P^\omega(\cdot)\mathbb{P}(d\omega)$ the distribution over everything: particles, walks and environments.

Make this moment assumption:

$$E[\eta_0(0)^2] < \infty. \tag{1.20}$$

Parameters that appear in the results are

$$\rho_0 = E[\eta_0(x)] \quad \text{and} \quad \sigma_0^2 = \mathbb{E}[\mathrm{Var}^\omega(\eta_0(0))]. \tag{1.21}$$

Next we describe the limiting process. Let $\dot{W}$ be space-time white noise corresponding and B a two-sided one-parameter Brownian motion on $\mathbb{R}$, independent of $\dot{W}$. Let W be the two-parameter Brownian motion on $\mathbb{R}_+ \times \mathbb{R}$ given by $W(t,r) = \dot{W}([0,t]\times[0,r])$, if $r > 0$, and $W(t,r) = -\dot{W}([0,t]\times[r,0])$, if $r < 0$. Define the process $Z(t,r)$ as the unique mild solution of the stochastic heat equation (see [25])

$$Z_t = \frac{\sigma^2}{2} Z_{rr} + \sqrt{\rho_0}\,\dot{W}, \qquad Z(0,r) = \sigma_0 B(r).$$

Process Z is given by

$$\begin{aligned} Z(t,r) = \sqrt{\rho_0} \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma^2(t-s)}(r-x)\,dW(s,x) \\ +\ \sigma_0 \int_{\mathbb{R}} \varphi_{\sigma^2 t}(r-x) B(x)\,dx, \end{aligned} \tag{1.22}$$

where $\varphi_{\nu^2}(x) = (2\pi\nu^2)^{-1/2}\exp(-x^2/2\nu^2)$ denotes the centered Gaussian density with variance ν^2, and $\Phi_{\nu^2}(x) = \int_{-\infty}^x \varphi_{\nu^2}(y)dy$ the distribution function.

$\{Z(t,r) : t \in \mathbb{R}_+, r \in \mathbb{R}\}$ is a mean zero Gaussian process. Its covariance can be expressed as follows: with

$$\Psi_{\nu^2}(x) = \nu^2 \varphi_{\nu^2}(x) - x\big(1 - \Phi_{\nu^2}(x)\big) \tag{1.23}$$

define two covariance functions on $(\mathbb{R}_+ \times \mathbb{R}) \times (\mathbb{R}_+ \times \mathbb{R})$ by

$$\Gamma_1\big((s,q),(t,r)\big) = \Psi_{\sigma^2(t+s)}(r-q) - \Psi_{\sigma^2|t-s|}(r-q) \tag{1.24}$$

and

$$\Gamma_2\big((s,q),(t,r)\big) = \Psi_{\sigma^2 s}(-q) + \Psi_{\sigma^2 t}(r) - \Psi_{\sigma^2(t+s)}(r-q). \tag{1.25}$$

Then

$$\mathbf{E}[Z(s,q)Z(t,r)] = \rho_0 \Gamma_1\big((s,q),(t,r)\big) + \sigma_0^2 \Gamma_2\big((s,q),(t,r)\big). \tag{1.26}$$

(Boldface **P** and **E** denote generic probabilities and expectations not connected with the RWRE model).

The theorem we state is for the finite-dimensional distributions of the current process, scaled and centered by its quenched mean:

$$\overline{Y}_n(t,r) = n^{-1/4}\big\{Y_n(t,r) - E^\omega[Y_n(t,r)]\big\}.$$

Fix any $N \in \mathbb{N}$, time points $0 < t_1 < t_2 < \cdots < t_N \in \mathbb{R}_+$, space points $r_1, r_2, \ldots, r_N \in \mathbb{R}$ and an N-vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$. Form the linear combinations

$$\overline{Y}_n(\boldsymbol{\theta}) = \sum_{i=1}^{N} \theta_i \overline{Y}_n(t_i, r_i) \quad \text{and} \quad Z(\boldsymbol{\theta}) = \sum_{i=1}^{N} \theta_i Z(t_i, r_i).$$

Theorem 1.4 *Consider independent particles on $\mathbb{Z}$ in an i.i.d. space-time environment with bounded jumps, under assumptions (1.3) and (1.5). Let the ω-dependent initial distribution satisfy (1.19) and (1.20). With definitions as above, quenched characteristic functions converge in $L^1(\mathbb{P})$:*

$$\lim_{n\to\infty} \mathbb{E}\left| E^{\omega}(e^{i\overline{Y}_n(\boldsymbol{\theta})}) - \mathbf{E}(e^{iZ(\boldsymbol{\theta})})\right| = 0. \tag{1.27}$$

In particular, under the averaged distribution P, convergence in distribution holds for the $\mathbb{R}^N$-valued vectors as $n \to \infty$:

$$\big(\overline{Y}_n(t_1, r_1), \overline{Y}_n(t_2, r_2), \cdots, \overline{Y}_n(t_N, r_N)\big) \Rightarrow \big(Z(t_1, r_1), Z(t_2, r_2), \cdots, Z(t_N, r_N)\big).$$

While we do not have a quenched limit (convergence of distributions under a fixed ω), limit (1.27) does imply that, if a quenched limit exists, it is the same as we have found.

A special case of the above theorem is the stationary situation. The proof of the following corollary comes by a direct computation using (1.26).

Corollary 1.5 *Consider the same setting as in the previous theorem. If furthermore variables η_0 have conditional distribution (1.6), and more generally when $\sigma_0^2 = \rho_0$, process $Z(t, 0)$ has covariance*

$$\mathbf{E}[Z(s, 0)Z(t, 0)] = \frac{\rho_0 \sigma}{\sqrt{2\pi}}(\sqrt{s} + \sqrt{t} - \sqrt{|t - s|}\,),$$

i.e. $\rho_0^{-1}\sigma^{-1}\sqrt{\pi/2}\, Z(t, 0)$ is a fractional Brownian motion with Hurst parameter $1/4$.

The next two theorems are on fluctuations of the quenched mean process $E^{\omega} Y_n(t, r)$ in the special case of one-dimensional random walks with admissible steps 0 and 1. Although we expect the result to hold for more general random walks, this is the only case for which we are able to characterize the fluctuations. Let $\sigma_D^2 = \mathbb{V}\mathrm{ar}(\omega_{0,0})$ and $\alpha = \mathbb{E}\omega_{0,0}(1 - \omega_{0,0})$. Note that $v = p(1) = \mathbb{E}\omega_{0,0}$ and $\sigma^2 = v(1 - v)$.

First, we consider the case of an initial configuration η_0 with independent quenched means.

Theorem 1.6 *Let* $\{\eta_0(x) : x \in \mathbb{Z}\}$ *be such that the quenched means* $\{E^\omega \eta_0(x) : x \in \mathbb{Z}\}$ *are independent with mean* ρ_0 *and variance* σ_0^2*. Assume that there exists* $\varepsilon > 0$ *such that* $\sup_x \mathbb{E}[|E^\omega \eta_0(x)|^{2+\varepsilon}] < \infty$*. Assume the* η_0*-variables are independent of the transition probabilities* $\{\omega_{x,t} : (x,t) \in \mathbb{Z} \times \mathbb{Z}_+\}$*. Then the finite-dimensional marginals of the process* $\{n^{-1/4} E^\omega\big(Y_n(t,r) - \rho_0 r\sqrt{n}\big) : t \geqslant 0, r \in \mathbb{R}\}$ *converge weakly as* $n \to \infty$ *to those of the unique mild solution to the stochastic heat equation*

$$z_t = \frac{\sigma^2}{2} z_{rr} + \frac{\rho_0 \sigma_D}{\sqrt{\alpha}}\, \dot{W}, \quad z(0,r) = \sigma_0 B(r),$$

where $\dot{W}$ *is space-time white noise and* B *a two-sided Brownian, independent of* $\dot{W}$*.*

The above includes the case when η_0 is independent of ω altogether. In that case, $\sigma_0 = 0$ and thus the initial condition becomes $z(0,r) \equiv 0$.

Next, we look at the stationary case. The reason this is different from the previous theorem is that now the quenched means of the initial occupation variables are not independent.

Theorem 1.7 *Let* $\{\eta_0(x) : x \in \mathbb{Z}^d\}$ *be distributed according to* (1.6) *with* $\rho = 1$*. Assume the averaged probabilities* $p_0 = p_1 = 1/2$ *so that* $v = 1/2$*. Then for* $t \geqslant s > 0$ *we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{C}\mathrm{ov}\big(E^\omega Y_n(s,0), E^\omega Y_n(t,0)\big) = \frac{1}{2\sqrt{2\pi}}\big(\tfrac{1}{4}\alpha^{-1} - 1\big)\big(\sqrt{t} + \sqrt{s} - \sqrt{t-s}\,\big).$$

The above limit matches the covariance structure of a constant $((\frac{1}{4}\alpha^{-1} - 1)^{1/2}/(2\pi)^{1/4})$ times a fractional Brownian motion with Hurst parameter $1/4$. Theorems 1.6 and 1.7 are proved in Sect. 6. At the end of that section, we explain why we expect the same limiting behavior in the setting of Theorem 1.7 as that of Theorem 1.6 and how this would imply the fractional Brownian motion limit.

Further notational conventions $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Multi-step transition probabilities from time s to time $t > s + 1$ are

$$\pi^\omega_{s,t}(x,y) = \sum_{u_1,\ldots,u_{t-s-1} \in \mathbb{Z}^d} \pi^\omega_{s,s+1}(x,u_1)\pi^\omega_{s+1,s+2}(u_1,u_2)\cdots \pi^\omega_{t-1,t}(u_{t-s-1},y).$$

We omit floor notation from time parameters, and so for the walk, $X_t = X_{\lfloor t \rfloor}$ for real $t \geqslant 0$. No jumps happen between integer times.

$\mathcal{S}$ denotes the set of all measures μ on $(\mathbb{Z}_+)^{\mathbb{Z}^d}$ that are invariant under spatial translations. $\mathcal{S}_e$ denotes the subset of $\mathcal{S}$ consisting of ergodic measures. $\mathcal{I}$ denotes the set of measures that are invariant for the particle evolution, that is $\mu_t = \mu S(t) = \mu$ for all $t \in \mathbb{Z}_+$ (μ_t and $\mu S(t)$ here denote the measure on configurations at time t when the initial measure on configurations is μ). $\mathbb{E}$, E^ω, E, E_η, $\mathbf{E}$ etc will denote expectations with respect to $\mathbb{P}$, P^ω, P, P_η, $\mathbf{P}$, etc. Variances and covariances are denoted similarly. Constants C can change from term to term.

2 Coupled Process

This section describes the coupling that will be used to prove Theorem 1.1. We couple two processes η_t and ζ_t so that matched particles move together forever, while unmatched particles move independently. To do this precisely, choose for each space-time point (x, t) a collection $\Xi_{x,t} = \{v_{x,t}^{0,j}, v_{x,t}^{+,j}, v_{x,t}^{-,j} : j \in \mathbb{N}\}$ of i.i.d. $\mathbb{Z}^d$-valued jump vectors from distribution $\omega_{x,t}$. Given initial configurations η_0 and ζ_0, perform the following actions. At each site x set

$$\xi_0(x) = \eta_0(x) \wedge \zeta_0(x), \beta_0^+(x) = (\eta_0(x) - \zeta_0(x))^+, \text{ and } \beta_0^-(x) = (\eta_0(x) - \zeta_0(x))^-.$$

$\xi_0(x)$ is the number of matched particles, while $\beta_0^{\pm}(x)$ count the unmatched $(+)$ and $(-)$ particles. Move particles from each site x as follows: the $\xi_0(x)$ matched particles jump to locations $x + v_{x,0}^{0,j}$ for $j = 1, \ldots, \xi_0(x)$, the $\beta_0^+(x)$ $(+)$ particles jump to locations $x + v_{x,0}^{+,j}$ for $j = 1, \ldots, \beta_0^+(x)$, and the $\beta_0^-(x)$ $(-)$ particles jump to locations $x + v_{x,0}^{-,j}$ for $j = 1, \ldots, \beta_0^-(x)$. After all jumps from all sites have been executed, match as many pairs of $(+)$ and $(-)$ particles at the same site as possible. This means that a $(+-)$ pair together at the same site merges to create a single ξ-particle at the same site. (For example, if after the jumps site y contains s ξ-particles, k $(+)$ particles and ℓ $(-)$ particles, then set $\xi_1(y) = s + k \wedge \ell$ and $\beta_1^{\pm}(y) = (k - \ell)^{\pm}$). Since particles are not labeled, it is immaterial which particular $(+)$ particle merges with a particular $(-)$ particle. When this is complete we have defined the state $(\xi_1(x), \beta_1^+(x), \beta_1^-(x))_{x \in \mathbb{Z}^d}$ at time $t = 1$. Then repeat, utilizing the jump variables for time $t = 1$. And so on.

This produces a joint process $(\xi_t, \beta_t^+, \beta_t^-)$ such that

$$\xi_t(x) = \eta_t(x) \wedge \zeta_t(x), \beta_t^+(x) = (\eta_t(x) - \zeta_t(x))^+, \text{ and } \beta_t^-(x) = (\eta_t(x) - \zeta_t(x))^-.$$

The η and ζ processes are recovered from

$$\eta_t(x) = \xi_t(x) + \beta_t^+(x) \quad \text{and} \quad \zeta_t(x) = \xi_t(x) + \beta_t^-(x).$$

The definition has the effect that a matched pair of η and ζ particles stays forever together, while a pair of $(+)$ and $(-)$ particles together at a site annihilate each other and turn into a matched pair. If we are only interested in the evolution of the discrepancies (β_t^+, β_t^-) we can discard all matched pairs as soon as they arise, and simply consider independently evolving $(+)$ and $(-)$ particles that annihilate each other upon meeting.

If we denote by $\Xi = \{\Xi_{x,t} : x \in \mathbb{Z}^d, t \in \mathbb{Z}\}$ the collection of jump variables, and by $G_{0,t}$ the function that constructs the values at the origin at time t:

$$\big(\xi_t(0), \beta_t^+(0), \beta_t^-(0)\big) = G_{0,t}(\eta_0, \zeta_0, \Xi)$$

then it is clear that the values at other sites x are constructed by applying this same function to shifted input:

$$\big(\xi_t(x), \beta_t^+(x), \beta_t^-(x)\big) = G_{0,t}(\theta_x\eta_0, \theta_x\zeta_0, \theta_x\Xi). \tag{2.1}$$

Here θ_x is a spatial shift: $(\theta_x\eta)(y) = \eta(x+y)$ and $(\theta_x\Xi)_{y,t} = \Xi_{x+y,t}$ for $x, y \in \mathbb{Z}^d$. In particular, if the initial distribution $\widetilde{\mu}$ of the pair (η_0, ζ_0) is invariant and ergodic under the shifts θ_x, while $\{\Xi_{x,t} : x \in \mathbb{Z}^d, t \in \mathbb{Z}\}$ are i.i.d. and independent of (η_0, ζ_0), it follows first that the triple (η_0, ζ_0, Ξ) is ergodic, and then from (2.1) that for each fixed t the configuration $(\xi_t, \beta_t^+, \beta_t^-)$ is invariant and ergodic under the shifts θ_x.

Let $\widetilde{\mathcal{S}}$, resp. $\widetilde{\mathcal{S}}_e$, denote the set of spatially invariant, resp. ergodic, probability distributions on pairs (η, ζ) of configurations of occupation variables.

Lemma 2.1 *Let $\widetilde{\mu} \in \widetilde{\mathcal{S}}$. The expectations $E_{\widetilde{\mu}}[\beta_t^+(x)]$ and $E_{\widetilde{\mu}}[\beta_t^-(x)]$ are independent of x and nonincreasing in t.*

Proof The independence of x is due to the shift-invariance from (2.1). That $E_{\widetilde{\mu}}[\beta_t^{\pm}(x)]$ is nonincreasing in t follows from the fact that discrepancy particles are not created, only annihilated. □

Proposition 2.2 *Let $d = 1$ or 2. Suppose $\widetilde{\mu} \in \widetilde{\mathcal{S}}_e$. Let $E_{\widetilde{\mu}}[\eta(0)] = \rho_1$ and $E_{\widetilde{\mu}}[\zeta(0)] = \rho_2$. If $\rho_1 \geqslant \rho_2$, we have*

$$E_{\widetilde{\mu}}[\beta_t^-(0)] = E_{\widetilde{\mu}}[(\eta_t(0) - \zeta_t(0))^-] \to 0 \text{ as } t \to \infty.$$

Proof We already know from Lemma 2.1 that $E_{\widetilde{\mu}}[\beta_t^{\pm}(0)]$ cannot increase. To get a contradiction let us assume that $E_{\widetilde{\mu}}[\beta_t^-(0)] \geqslant \delta$ for all t and some $\delta > 0$. Since $E_{\widetilde{\mu}}[\beta_t^+(0)] - E_{\widetilde{\mu}}[\beta_t^-(0)] = E_{\widetilde{\mu}}[\eta_t(0)] - E_{\widetilde{\mu}}[\zeta_t(0)] = \rho_1 - \rho_2 \geqslant 0$, we also have $E_{\widetilde{\mu}}[\beta_t^+(0)] \geqslant \delta$ for all t.

At time 0, assign labels separately to the $(+)$ and $(-)$ particles from some countable label sets $\mathcal{J}^+$ and $\mathcal{J}^-$ and denote the locations of these particles by $\{w_i^+(t), w_j^-(t) : i \in \mathcal{J}^+, j \in \mathcal{J}^-\}$. Each $(+)$ and $(-)$ particle retains its label throughout its lifetime. The lifetime of $(+)$ particle $j \in \mathcal{J}^+$ is

$$\tau_j^+ = \inf\{t \geqslant 0 : w_j^+ \text{ is annihilated by a } (-) \text{ particle}\}.$$

If $\tau_j^+ = \infty$, then j is *immortal*. Similarly define τ_j^-. Let

$$\beta_{0,t}^{\pm}(x) = \sum_j \mathbf{1}\{w_j^{\pm}(0) = x, \tau_j^{\pm} > t\}$$

denote the number of $(\pm)$ particles initially at site x that live past time t. We would like to claim that for a fixed t the configuration $\{(\beta_{0,t}^+(x), \beta_{0,t}^-(x)) : x \in \mathbb{Z}^d\}$ is invariant and ergodic under the spatial shifts θ_x. This will be true if the evolution is given by a mapping $F_{0,t}$ so that $(\beta_{0,t}^+(x), \beta_{0,t}^-(x)) = F_{0,t}(\theta_x\eta_0, \theta_x\zeta_0, \theta_x\Xi)$ for all $x \in \mathbb{Z}^d$. Such a mapping can be created by specifying precise rules for the movement and annihilation of $(+)$ and $(-)$ particles that are naturally invariant under shifts. For example, we can take $\mathcal{J}^{\pm} \subset \mathbb{Z}$ and give the sites of $\mathbb{Z}^d$ some ordering. Label particles initially in increasing order, so that $i < j$ implies $w_i^{\pm}(0) \leqslant w_j^{\pm}(0)$. Then at each time step particles from a given site are distributed to their subsequent locations in increasing order, and $(+, -)$ pairs are matched beginning with lowest labels. Of course the overall ordering of particles is not preserved, but this mechanism does not depend on the absolute labels, only their ordering, and respects the spatial translations.

Then the ergodic theorem implies that

$$E_{\widetilde{\mu}}[\beta^{\pm}_{0,t}(0)] = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|\leqslant n} \beta^{\pm}_{0,t}(x) \text{ a.s.}$$

Here $|x|$ is the ℓ^∞ norm: for a vector $x = (x_1, \ldots, x_d)$, $|x| = \max_{1\leqslant i\leqslant d} |x_i|$. Since particles take jumps of magnitude at most R,

$$\begin{aligned}\delta \leqslant E_{\widetilde{\mu}}[\beta^{\pm}_{t}(0)] &= \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|\leqslant n} \beta^{\pm}_{t}(x)\\ &\leqslant \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|\leqslant n+Rt} \beta^{\pm}_{0,t}(x) = E_{\widetilde{\mu}}[\beta^{\pm}_{0,t}(0)].\end{aligned}$$

The initial occupation numbers of immortal $+/-$ particles are

$$\beta^{\pm}_{0,\infty}(x) = \lim_{t\to\infty} \beta^{\pm}_{0,t}(x).$$

The limit exists by monotonicity. This limit produces again a functional relationship of the type (2.1):

$$\begin{aligned}\big(\beta^{+}_{0,\infty}(x), \beta^{-}_{0,\infty}(x)\big) &= \lim_{t\to\infty} \big(\beta^{+}_{0,t}(x), \beta^{-}_{0,t}(x)\big) = \lim_{t\to\infty} F_{0,t}(\theta_x\eta_0, \theta_x\zeta_0, \theta_x\,\Xi)\\ &= F_{0,\infty}(\theta_x\eta_0, \theta_x\zeta_0, \theta_x\,\Xi).\end{aligned}$$

Thereby $\{(\beta^{+}_{0,\infty}(x), \beta^{-}_{0,\infty}(x)) : x \in \mathbb{Z}^d\}$ is spatially invariant and ergodic.

By the ergodic theorem again

$$E_{\widetilde{\mu}}[\beta^{\pm}_{0,\infty}(0)] = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|\leqslant n} \beta^{\pm}_{0,\infty}(x) \quad \text{a.s.}$$

while by the monotone convergence theorem

$$E_{\widetilde{\mu}}[\beta^{\pm}_{0,\infty}(0)] = \lim_{t\to\infty} E_{\widetilde{\mu}}[\beta^{\pm}_{0,t}(0)] \geqslant \delta.$$

We have shown that the assumption $E_{\widetilde{\mu}}[\beta_t^-(0)] \geqslant \delta$ leads to the existence of positive densities of immortal $(+)$ and $(-)$ particles. However, a situation like this will never arise for $d = 1$ or 2, the reason being that any two particles on the lattice will meet each other infinitely often. More precisely, fix any two particles and let $X_\bullet^+$ and $X_\bullet^-$ denote the walks undertaken by these two particles. Then $X_\bullet^+$ and $X_\bullet^-$ are two independent walks in a common environment ω. Let $Y_t = X_t^+ - X_t^-$. If we average out the environment, then Y_t is a Markov chain on $\mathbb{Z}^d$ with transition $q(x, y)$ given by (1.8). Away from the origin this is a symmetric random walk with bounded steps, and hence recurrent when $d = 1$ or 2. Thus $Y_t = 0$ infinitely often. We have arrived at a contradiction and the proposition is proved. □

3 Invariant Measures

In this section we prove Theorem 1.1. We begin by deriving the well-known invariant density for the environment process seen by a single tagged particle.

Proposition 3.1 *There exists a function* $0 \leqslant f < \infty$ *on* Ω *such that* $\mathbb{E} f = 1$, $\mathbb{E}(f^2) < \infty$, $f(\omega)$ *is a function of* $\bar{\omega}_{-\infty,-1}$, *and*

$$f(\omega) = \sum_{x \in \mathbb{Z}^d} f(T_{x,-1}\omega)\pi^{\omega}_{-1,0}(x, 0) \quad \mathbb{P}\text{-almost surely.} \tag{3.1}$$

Proof For $N \in \mathbb{Z}_+$ define

$$f_N(\omega) = \sum_{z \in \mathbb{Z}^d} \pi^{\omega}_{-N,0}(z, 0). \tag{3.2}$$

$f_N(\omega)$ is $\mathfrak{S}_{-N,-1}$-measurable and a martingale with $\mathbb{E} f_N = 1$. By the martingale convergence theorem we can define

$$f(\omega) = \lim_{N\to\infty} f_N(\omega) \quad (\mathbb{P}\text{-almost sure limit}).$$

Property (3.1) follows because all the sums involved are finite:

$$\begin{aligned}
&\sum_x f(T_{x,-1}\omega)\pi^{\omega}_{-1,0}(x, 0) = \lim_{N\to\infty} \sum_x f_N(T_{x,-1}\omega)\pi^{\omega}_{-1,0}(x, 0) \\
&= \lim_{N\to\infty} \sum_{z,x} \pi^{\omega}_{-N-1,-1}(z, x)\pi^{\omega}_{-1,0}(x, 0) = \lim_{N\to\infty} \sum_z \pi^{\omega}_{-N-1,0}(z, 0) \\
&= \lim_{N\to\infty} f_{N+1}(\omega) = f(\omega).
\end{aligned}$$

In Lemma 3.4 below we show the L^2 boundedness of the sequence $\{f_N\}$. This implies that $f_N \to f$ also in L^2 and thereby implies the remaining statements $\mathbb{E}f = 1$ and $\mathbb{E}(f^2) < \infty$. □

The following addresses the positivity of f.

Lemma 3.2 $\mathbb{P}(f > 0) = 1$ *if and only if there exists an* x *such that* $\mathbb{P}\{\pi_{0,1}(0, x) > 0\} = 1$.

Proof If there does not exist an x as in the claim, then by independence of the environment and the finite step-size assumption we see that

$$\mathbb{P}\{\forall x : \pi_{-1,0}(x, 0) = 0\} > 0.$$

But then (3.1) implies that $\mathbb{P}(f = 0) > 0$. Conversely, if there exists an x as in the claim, then (3.1) implies that if $f(\omega) = 0$ then $f(T_{x,-1}\omega) = 0$. Shift-invariance implies the two events are in fact equal, almost surely. This in turn implies that $\{f = 0\}$ is a trivial event and since $\mathbb{E}[f] = 1$ we have that $f > 0$ a.s. □

To prove the L^2 estimate for f_N we develop a Green function bound for the Markov chain defined as the difference of two walks. Let X_t^x and $\widetilde{X}_t^y$ be two independent walks in a common environment ω, started at $x, y \in \mathbb{Z}^d$, and $Y_t = X_t^x - \widetilde{X}_t^y$. Under the averaged measure Y_t is a Markov chain on $\mathbb{Z}^d$ with transition probabilities $q(x, y)$ defined by (1.8). Y_t can be thought of as a symmetric random walk on $\mathbb{Z}^d$ whose transition has been perturbed at the origin. The corresponding homogeneous, unperturbed random walk is $\bar{Y}_t$ with transition probability $\bar{q}$ in (1.9). Write P_x and $\bar{P}_x$ for the path probabilities of $Y_\bullet$ and $\bar{Y}_\bullet$. Define hitting times of 0 for both walks Y_t and $\bar{Y}_t$ by

$$\tau = \inf\{n \geqslant 1 : Y_n = 0\} \quad \text{and} \quad \bar{\tau} = \inf\{n \geqslant 1 : \bar{Y}_n = 0\}. \tag{3.3}$$

Denote the k-step transition probabilities by $q^k(x, y)$ and $\bar{q}^k(x, y)$.

Lemma 3.3 *There exists a constant* $C < \infty$ *such that for all* $x \in \mathbb{Z}^d$ *and* $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} q^k(x, 0) \leqslant C \sum_{k=0}^{N} \bar{q}^k(x, 0).$$

Proof Suppose we had the bound for $x = 0$. Then it follows for $x \neq 0$:

$$\begin{aligned}\sum_{k=0}^{N} q^k(x,0) &= E_x\Big[\sum_{k=0}^{N} \mathbf{1}_{\{Y_k=0\}}\Big] = E_x\Big[\sum_{i=0}^{N} \mathbf{1}_{\{\tau=i\}} \sum_{k=i}^{N} \mathbf{1}_{\{Y_k=0\}}\Big] \\ &= \sum_{i=0}^{N} P_x(\tau=i) \sum_{k=0}^{n-i} q^k(0,0) \leqslant C \sum_{i=0}^{N} \bar{P}_x(\bar{\tau}=i) \sum_{k=0}^{n-i} \bar{q}^k(0,0) \\ &= C \sum_{k=0}^{N} \bar{q}^k(x,0)\end{aligned}$$

It remains to prove the result for $x = 0$. Let $\sigma_0 = 0$ and

$$\sigma_{j+1} = \inf\{n > \sigma_j : Y_n = 0 \text{ and } Y_k \neq 0 \text{ for some } k \in \{\sigma_j + 1, \dots, n-1\}\}.$$

These are the successive times of arrivals to 0 following excursions away from 0. Let W_j, $j \geq 0$, be the durations of the sojourns at 0, in other words

$$Y_n = 0 \text{ iff } \sigma_j \leq n < \sigma_j + W_j \text{ for some } j \geq 0.$$

Sojourns are geometric and independent of the past, so on the event $\{\sigma_j < \infty\}$,

$$E_0(W_j \mid \mathcal{F}^Y_{\sigma_j}) = \frac{1}{1 - q(0,0)}.$$

Let $J_N = \max\{j \geq 0 : \sigma_j \leq N\}$ mark the last sojourn at 0 that started by time N. Then

$$\begin{aligned}E_0\Big[\sum_{k=0}^{N} \mathbf{1}\{Y_k = 0\}\Big] &\leq E_0\Big[\sum_{j=0}^{J_N} W_j\Big] = \sum_{j=0}^{\infty} E_0\big[\mathbf{1}\{\sigma_j \leq N\} W_j\big] \\ &= \frac{1}{1-q(0,0)} E_0(1 + J_N).\end{aligned}$$

Assumption (1.4) guarantees that $q(0,0) < 1$.

It remains to bound $E_0(1 + J_N)$ in terms of $\sum_{k=0}^{N} \bar{q}^k(0,0)$. The key is that once the Markov chain Y_k has left the origin, it follows the same transitions as the homogeneous walk $\bar{Y}_k$ until the next visit to 0. For $z \neq 0$ let

$$K^z_N = \inf\{k \geqslant 1 : T^z_1 + T^z_2 + \cdots + T^z_k \geqslant N\}$$

where the $\{T^z_i\}$ are i.i.d. with common distribution $\bar{P}_z\{\bar{\tau} \in \cdot\}$. Imagine constructing the path Y_k so that every step away from 0 is followed by an excursion of $\bar{Y}_k$ that ends at 0 (or continues forever if 0 is never reached). The step bound (1.1) implies that $P_0\{|Y_1| \leqslant 2R\} = 1$. Then there is stochastic dominance that gives

$$E_0(J_N) \leqslant \sum_{z \neq 0\,:\,|z| \leqslant 2R} \bar{E}(K_N^z). \tag{3.4}$$

By T32.1 in Spitzer [22, p. 378], for $z \neq 0$

$$\lim_{n\to\infty} \frac{\bar{P}_z(\bar{\tau} > n)}{\bar{P}_0(\bar{\tau} > n)} = \bar{a}(z) \tag{3.5}$$

where the potential kernel $\bar{a}$ is

$$\bar{a}(z) = \lim_{n\to\infty} \left\{ \sum_{k=0}^{n} \bar{q}^k(0,0) - \sum_{k=0}^{n} \bar{q}^k(z,0) \right\}. \tag{3.6}$$

By P30.2 in [22, p. 361] $\bar{a}(z) > 0$ for all $z \neq 0$ for $d = 1, 2$. For $d \geqslant 3$ by transience

$$\bar{a}(z) = \sum_{k=0}^{\infty} \bar{q}^k(0,0) - \sum_{k=0}^{\infty} \bar{q}^k(z,0) = (1 - \bar{F}(z,0)) \sum_{k=0}^{\infty} \bar{q}^k(0,0) > 0$$

where $\bar{F}(z,0) = \bar{P}_z\{\bar{Y}_n = 0 \text{ for some } n \geqslant 1\} < 1$.

From (3.5) and $\bar{a}(z) > 0$, there exist $0 < c(z), C(z) < \infty$ such that for all n,

$$c(z)\bar{P}_0(\bar{\tau} > n) \leqslant \bar{P}(T_1^z > n) \leqslant C(z)\bar{P}_0(\bar{\tau} > n)$$

and hence

$$c(z)\bar{E}_0\big[\bar{\tau} \wedge N\big] \leqslant \bar{E}\big[T_1^z \wedge N\big] \leqslant C(z)\bar{E}_0\big[\bar{\tau} \wedge N\big].$$

By Wald's identity and some simple bounds (see Exercice 4.4.1 in [6, Sect. 4.4])

$$\frac{N}{\bar{E}\big(T_1^z \wedge N\big)} \leqslant \bar{E}\big[K_N^z\big] \leqslant \frac{2N}{\bar{E}\big(T_1^z \wedge N\big)}.$$

Let $\{\bar{\tau}_i\}$ be i.i.d. copies of $\bar{\tau}$ from (3.3) and put

$$M_N = \inf\{k \geqslant 1 : \bar{\tau}_1 + \bar{\tau}_2 + \cdots + \bar{\tau}_k \geqslant N\}.$$

Then we have a similar relation:

$$\frac{N}{\bar{E}_0\big(\bar{\tau} \wedge N\big)} \leqslant \bar{E}_0\big[M_N\big] \leqslant \frac{2N}{\bar{E}_0\big(\bar{\tau} \wedge N\big)}$$

Combining the above lines:

$$\bar{E}\big[K_N^z\big] \leqslant \frac{2N}{\bar{E}\big[T_1^z \wedge N\big]} \leqslant \frac{2N}{c(z)\bar{E}_0\big[\bar{\tau} \wedge N\big]} \leqslant \frac{2}{c(z)}\bar{E}_0\big[M_N\big]. \tag{3.7}$$

Considering excursions of the $\bar{Y}$-walk away from 0,

$$\bar{E}_0\big[M_N\big] \leqslant 1 + \sum_{k=0}^{N} \bar{q}^k(0,0) \leqslant 2\sum_{k=0}^{N} \bar{q}^k(0,0). \tag{3.8}$$

The proof is now complete with a combination of (3.4), (3.7) and (3.8). □

From the previous lemma follows the L^2 estimate for f_N which completes the proof of Proposition 3.1.

Lemma 3.4 *There exists a constant $C < \infty$ such that $\mathbb{E}(f_N^2) \leqslant C$ for all N.*

Proof By translations

$$\begin{aligned}\mathbb{E}(f_N^2) &= \sum_{x,z} \mathbb{E}\pi^{\omega}_{-N,0}(x,0)\pi^{\omega}_{-N,0}(z,0)\\ &= \sum_{y} \mathbb{E}P^{\omega}\{X_N^y = \widetilde{X}_N^0\} = \sum_{y} q^N(y,0).\end{aligned}$$

By the submartingale property $\mathbb{E}(f_N^2)$ is nondecreasing in N. Hence it suffices to show the existence of a constant C such that

$$\sum_{k=0}^{N} \mathbb{E}(f_k^2) \leqslant C(N+1) \qquad \text{for all } N. \tag{3.9}$$

From above, by Lemma 3.3 and the spatial homogeneity of the $\bar{Y}$-walk,

$$\begin{aligned}\sum_{k=0}^{N} \mathbb{E}(f_k^2) &= \sum_{k=0}^{N}\sum_{x} q^k(x,0) \leqslant C\sum_{k=0}^{N}\sum_{x} \bar{q}^k(x,0)\\ &= C\sum_{k=0}^{N}\sum_{x} \bar{q}^k(0,x) = C(N+1).\end{aligned}$$ □

Property (3.1) implies that the probability measure $\mathbb{P}_\infty(d\omega) = f(\omega)\,\mathbb{P}(d\omega)$ is invariant for the process $T_{X_n,n}\omega$. Recall from (1.6) the product measure

$$\mu^{\rho,\omega}(d\eta) = \bigotimes_{x\in\mathbb{Z}^d} \Gamma^{\rho f(T_{x,0}\omega)}\big(d\eta(x)\big) \tag{3.10}$$

where Γ^λ is Poisson(λ) distribution. By the definition of f, $\mu^{\rho,\omega}$ depends on ω only through the levels $\bar{\omega}_{-\infty,-1}$.

Lemma 3.5 *The following holds for $\mathbb{P}$-a.e. ω. Let η_0 be $\mu^{\rho,\omega}$-distributed. Then for all times $t \in \mathbb{Z}_+$, under the evolution in the environment ω, η_t is $\mu^{\rho,T_{0,t}\omega}$-distributed, and in particular independent of the environment $\bar{\omega}_t$ at level t.*

Proof Consider the evolution under a fixed ω. The claim made in the lemma is true at time $t = 0$ by the construction. Suppose it is true up to time $t-1$. Then over $x \in \mathbb{Z}^d$ the variables $\eta_{t-1}(x)$ are independent Poisson variables with means $\rho f(T_{x,t-1}\omega)$. Each particle at site x chooses its next position y independently with probabilities $\pi^{\omega}_{t-1,t}(x,y)$. As with marking a Poisson process with independent coin flips, the consequence is that the numbers of particles going from x to y are independent Poisson variables with means $\rho f(T_{x,t-1}\omega)\pi^{\omega}_{t-1,t}(x,y)$, over all pairs (x,y). Since sums of independent Poissons are Poisson, the variables $(\eta_t(y))_{y\in\mathbb{Z}}$ are again independent Poissons and $\eta_t(y)$ has mean

$$\sum_x \rho f(T_{x,t-1}\omega)\pi^{\omega}_{t-1,t}(x,y) = \sum_z \rho f(T_{z,-1}T_{y,t}\omega)\pi^{T_{y,t}\omega}_{-1,0}(z,0)$$
$$= \rho f(T_{y,t}\omega).$$

The last equality is from (3.1).

We have shown that $\eta_t = (\eta_t(y))_{y\in\mathbb{Z}^d}$ has distribution $\mu^{\rho,\,T_{0,t}\omega}$. This measure is a function of $\bar{\omega}_{-\infty,t-1}$, hence independent of $\bar{\omega}_t$ under $\mathbb{P}$. □

Recall from (1.7) the averaged measure $\mu^{\rho} = \int \mu^{\rho,\omega}\,\mathbb{P}(d\omega)$.

Lemma 3.6 *The measure μ^{ρ} is invariant and ergodic under spatial shifts θ_x. The tail σ-field of the state space $\mathbb{Z}_+^{\mathbb{Z}^d}$ is trivial under μ^{ρ}.*

Proof Invariance under θ_x comes from $\mu^{\rho,\omega}\circ\theta_x^{-1} = \mu^{\rho,T_{x,0}\omega}$ and the invariance of $\mathbb{P}$. Ergodicity will follow from tail triviality.

Let $B \subseteq \mathbb{Z}_+^{\mathbb{Z}^d}$ be a tail event. Then by Kolmogorov's 0-1 law $\mu^{\rho,\omega}(B) \in \{0,1\}$ for each ω. We need to show that $\mu^{\rho,\omega}(B)$ is $\mathbb{P}$-a.s. constant. For this it suffices to show that $\mu^{\rho,\omega}(B)$ is itself (almost surely) a tail measurable function of ω.

Consider a ball $\Lambda = \{(z,s) : |s| + |z| \leqslant M\}$ in the space-time lattice $\mathbb{Z}^d \times \mathbb{Z}$. Since the step size of the walks is bounded by R, for each $x \in \mathbb{Z}^d$ and $N \geqslant 1$

$$f_N(T_{x,0}\omega) = \sum_{z\in\mathbb{Z}^d} \pi^{\omega}_{-N,0}(z,x)$$

is a function of the environments $\{\omega_{z,s} : s \leqslant -1, |z-x| \leqslant R|s|\}$. Consequently, if $|x| > (R+1)M$, the entire sequence $\{f_N(T_{x,0}\omega)\}_{N\in\mathbb{N}}$ is a function of the environments outside Λ, and then so is (almost surely) the limit $f(T_{x,0}\omega)$. Since B is tail measurable, $\mu^{\rho,\omega}(B)$ is a function of $\{f(T_{x,0}\omega) : |x| > (R+1)M\}$ and thereby a function of environments outside Λ. Since Λ was arbitrary, we conclude that $\mu^{\rho,\omega}(B)$ is (almost surely) a tail measurable function of ω. □

Proof of the first part of Theorem 1.1 (*except for uniqueness*) Invariance of μ^{ρ} for the process follows by averaging out ω in the result of Lemma 3.5. Spatial invariance, ergodicity and tail triviality of μ^{ρ} are in Lemma 3.6. That $\int \eta(0)\,d\mu^{\rho} = \rho$ follows from the definition of μ^{ρ}.

We prove the ergodicity of the process $\eta_\cdot$ under the time-shift-invariant path measure P_{μ^ρ}. We use the notation μ^ρ also for the joint measure $\mu^\rho(d\omega, d\eta) = \mathbb{P}(d\omega)\mu^{\rho,\omega}(d\eta)$ and not only for the marginal on η. Let $\mathcal{J}$ be the σ-algebra of invariant sets on the state space of the particle system:

$$\mathcal{J} = \{B \subseteq \mathbb{Z}_+^{\mathbb{Z}^d} : \mathbf{1}_B(\eta) = P_\eta\{\eta_1 \in B\} \text{ for } \mu^\rho\text{-a.s. } \eta\}$$

By Corollary 5 on p. 97 of [17] it suffices to show that $\mathcal{J}$ is trivial. We establish triviality of $\mathcal{J}$ by showing that $E^{\mu^\rho}[\psi \mid \mathcal{J}]$ is almost surely a constant for an arbitrary bounded cylinder function ψ on $\mathbb{Z}_+^{\mathbb{Z}^d}$.

Let $\eta^{a,b}$ denote the configuration obtained by moving one particle from site a to site b, if possible: $\eta^{a,b} = \eta$ if $\eta(a) = 0$, while if $\eta(a) > 0$,

$$\eta^{a,b}(x) = \begin{cases} \eta(a) - 1 & x = a \\ \eta(b) + 1 & x = b \\ \eta(x) & x \neq a, b. \end{cases}$$

Lemma 3.7 *There exists a version of $E^{\mu^\rho}[\psi \mid \mathcal{J}]$ such that for all $\eta \in \mathbb{Z}_+^{\mathbb{Z}^d}$ and $a, b \in \mathbb{Z}^d$, $E^{\mu^\rho}[\psi \mid \mathcal{J}](\eta) = E^{\mu^\rho}[\psi \mid \mathcal{J}](\eta^{a,b})$.*

Proof By Corollary 2 on p. 93 of [17], we can define a version $\widetilde{\psi}$ of $E^{\mu^\rho}[\psi \mid \mathcal{J}]$ pointwise by

$$\widetilde{\psi}(\eta) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{t=0}^{n-1} E_\eta[\psi(\eta_t)].$$

We show that $\widetilde{\psi}(\eta) = \widetilde{\psi}(\eta^{a,b})$.

Assume $\eta(a) > 0$. Consider the basic coupling $P_{\eta,\eta^{a,b}}$ of two processes (η_t, ζ_t) with initial configurations $(\eta_0, \zeta_0) = (\eta, \eta^{a,b})$, as described in Sect. 2. Let

$$\sigma = \inf\{t : \psi(\eta_s) = \psi(\zeta_s) \text{ for all } s \geqslant t\}.$$

We observe that $P_{\eta,\eta^{a,b}}\{\sigma < \infty\} = 1$ in all dimensions. In dimensions $d \in \{1, 2\}$ the irreducible $\bar{q}$-random walk is recurrent, hence the two discrepancies of opposite sign that start at a and b annihilate with probability 1. In dimensions $d \geqslant 3$ the discrepancies are marginally genuinely d-dimensional random walks by assumption (1.5). Thus they are transient, and so either the discrepancies annihilate or eventually they never return to the finite set of sites that support ψ.

The conclusion of the lemma follows:

$$|E_\eta[\psi(\eta_t)] - E_{\eta^{a,b}}[\psi(\eta_t)]| \leqslant 2\|\psi\|_\infty P_{\eta,\eta^{a,b}}\{\sigma > t\} \longrightarrow 0 \quad \text{as } t \to \infty. \qquad \square$$

Lemma 3.8 *Suppose h is a bounded measurable function on $\mathbb{Z}_+^{\mathbb{Z}^d}$ such that for all $a, b \in \mathbb{Z}^d$, $h(\eta^{a,b}) = h(\eta)$ μ^ρ-a.s. Then there exists a tail measurable function h_1 such that $h = h_1$ μ^ρ-a.s.*

Proof To show approximate tail measurability we approximate by a cylinder function and then move particles far enough one by one. (We learned this trick from [21]). Let η^a denote the configuration obtained by removing one particle from site a if possible:

$$\eta^a(x) = \begin{cases} (\eta(a) - 1)^+ & x = a \\ \eta(x) & x \neq a. \end{cases}$$

Let $\varepsilon > 0$. Pick a bounded cylinder function $\tilde{h}$ such that $E^{\mu^\rho}|h - \tilde{h}|^2 < \varepsilon^2$. For each ω pick $b(\omega) \in \mathbb{Z}^d$ so that $f(T_{b(\omega),0}\omega) \geqslant 1/4$ and $\tilde{h}$ does not depend on the coordinate $\eta(b(\omega))$. Such $b(\omega)$ exists a.s. by the ergodic theorem since $\mathbb{E}f = 1$. Choose $b(\omega)$ so that it is a measurable function. Since $h(\eta) = h(\eta^{a,b(\omega)})$ $\mu^\rho(d\omega, d\eta)$-a.s. and $\tilde{h}(\eta^a) = \tilde{h}(\eta^{a,b(\omega)})$ by choice of $b(\omega)$,

$$\int |h(\eta) - h(\eta^a)|\, \mu^\rho(d\eta) \leqslant \int \mathbf{1}_{\{\eta(a)>0\}} |h(\eta^{a,b(\omega)}) - \tilde{h}(\eta^{a,b(\omega)})|\, \mu^\rho(d\omega, d\eta) + \int \mathbf{1}_{\{\eta(a)>0\}} |\tilde{h}(\eta^a) - h(\eta^a)|\, \mu^\rho(d\eta).$$

In the next calculation we bound the first integral after the inequality. Write $\eta = (\eta', \eta(a), \eta(b(\omega)))$ to make the coordinates at a and $b(\omega)$ explicit. Change summation indices and apply Cauchy-Schwarz:

$$\begin{aligned}
&\int \mathbf{1}_{\{\eta(a)>0\}} |h(\eta^{a,b(\omega)}) - \tilde{h}(\eta^{a,b(\omega)})|\, \mu^\rho(d\omega, d\eta) \\
&= \mathbb{E} \sum_{\substack{k>0 \\ \ell \geqslant 0}} \Gamma^{\rho f(T_{a,0}\omega)}(k) \Gamma^{\rho f(T_{b(\omega),0}\omega)}(\ell) \int |h(\eta', k-1, \ell+1) \\
&\qquad\qquad - \tilde{h}(\eta', k-1, \ell+1)|\, \mu^{\rho,\omega}(d\eta') \\
&= \mathbb{E} \sum_{\substack{m \geqslant 0 \\ n>0}} \frac{f(T_{a,0}\omega)}{m+1} \cdot \frac{n}{f(T_{b(\omega),0}\omega)} \cdot \Gamma^{\rho f(T_{a,0}\omega)}(m) \Gamma^{\rho f(T_{b(\omega),0}\omega)}(n) \\
&\qquad\qquad \times \int |h(\eta', m, n) - \tilde{h}(\eta', m, n)|\, \mu^{\rho,\omega}(d\eta') \\
&= \int \mathbf{1}_{\{\eta(b(\omega))>0\}} \frac{f(T_{a,0}\omega)}{\eta(a)+1} \cdot \frac{\eta(b(\omega))}{f(T_{b(\omega),0}\omega)} \cdot |h(\eta) - \tilde{h}(\eta)|\, \mu^\rho(d\omega, d\eta) \\
&\leqslant \left\{ \mathbb{E} \sum_{\substack{m \geqslant 0 \\ n>0}} \frac{f(T_{a,0}\omega)^2}{(m+1)^2} \cdot \frac{n^2}{f(T_{b(\omega),0}\omega)^2} \cdot \Gamma^{\rho f(T_{a,0}\omega)}(m) \Gamma^{\rho f(T_{b(\omega),0}\omega)}(n) \right\}^{1/2} \\
&\qquad\qquad \times \left\{ E^{\mu^\rho} |h - \tilde{h}|^2 \right\}^{1/2} \\
&\leqslant \sqrt{5\mathbb{E}[f^2]}\varepsilon.
\end{aligned}$$

To obtain the second equality we replace $\Gamma^{\rho f(T_{a,0}\omega)}(k)$ by $\Gamma^{\rho f(T_{a,0}\omega)}(k-1)\cdot\frac{f(T_{a,0}\omega)}{k}$ and similarly for $\Gamma^{\rho f(T_{b(\omega),0}\omega)}(l)$.

An analogous argument (but easier since we do not need the $b(\omega)$) gives

$$\int \mathbf{1}_{\{\eta(a)>0\}}|\tilde{h}(\eta^a) - h(\eta^a)|\,\mu^\rho(d\eta) \leqslant C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $h(\eta) = h(\eta^a)$ μ^ρ-a.s.

For given finite $\Lambda \subseteq \mathbb{Z}^d$, applying the mapping $\eta \mapsto \eta^a$ repeatedly to remove all particles from Λ shows that h equals a.s. a function g_Λ that does not depend on $(\eta(x) : x \in \Lambda)$. As $\Lambda \nearrow \mathbb{Z}^d$ along cubes, the limit $h_1 = \lim g_\Lambda$ exists a.s. by martingale convergence and is tail measurable. □

We can now conclude the proof of (temporal) ergodicity of the process $\eta_\cdot$. Lemmas 3.7 and 3.8 show that $E^{\mu^\rho}[\psi \mid \mathcal{J}]$ is μ^ρ-a.s. tail measurable, and hence a constant by Lemma 3.6. □

3.1 Proof of Uniqueness

In this subsection, we complete the proof of part (a) of Theorem 1.1 by showing that μ^ρ is the *unique* invariant distribution with the stated properties. We also prove the second part of Theorem 1.1. The proof of uniqueness uses standard techniques of interacting particle systems [11]. We will arrive at the proof of uniqueness through a sequence of lemmas.

For two configurations η, ζ of occupation variables, we say that $\eta \leqslant \zeta$ if $\eta(x) \leqslant \zeta(x)$ for all x. For two probability distributions μ, ν on the configuration space, we say $\mu \leqslant \nu$ if there exists a probability measure $\widetilde{\mu}$ on pairs (η, ζ) of configurations of occupation variables such that $\widetilde{\mu}(\eta \leqslant \zeta) = 1$ and the marginals of $\widetilde{\mu}$ are μ and ν. For a convex set $\mathcal{A}$, $\mathcal{A}_e$ will denote the set of extremal elements.

Recall that $\widetilde{\mathcal{S}}$, resp. $\widetilde{\mathcal{S}}_e$, denotes the set of spatially invariant resp. ergodic probability distributions on pairs (η, ζ) of configurations of occupation variables. Let $\tilde{\mathcal{I}}$ denote the set of probability distributions on pairs of configurations of occupation variables, that are invariant under the temporal evolution described at the beginning of Sect. 2.

Lemma 3.9 *If* $\rho_1 < \rho_2$ *then* $\mu^{\rho_1} \leqslant \mu^{\rho_2}$.

Proof We couple $\mu^{\rho_1,\omega}$ and $\mu^{\rho_2,\omega}$ by letting $\widetilde{\mu}^\omega$ be the distribution of (η, ζ) defined by letting occupation variables $\eta(x)$ be independent Poisson with means $\rho_1 f(T_{x,0}\omega)$, $\gamma(x)$ be independent Poisson with means $(\rho_2 - \rho_1) f(T_{x,0}\omega)$, and then setting $\zeta(x) = \eta(x) + \gamma(x)$. Then define the coupling of μ^{ρ_1} and μ^{ρ_2} by $\widetilde{\mu}(\cdot) = \mathbb{E}[\widetilde{\mu}^\omega(\cdot)]$. □

We state the next two lemmas without proof. The proofs can be found in Lemmas 4.2 - 4.5 of [1].

Lemma 3.10 *We have*

(a) *If* $\mu_1, \mu_2 \in \mathcal{I} \cap \mathcal{S}$, *there is a* $\widetilde{\mu} \in \widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}}$ *with marginals* μ_1 *and* μ_2.
(b) *If* $\mu_1, \mu_2 \in (\mathcal{I} \cap \mathcal{S})_e$, *there is a* $\widetilde{\mu} \in (\widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}})_e$ *with marginals* μ_1 *and* μ_2.

Lemma 3.11 *If* $\widetilde{\mu} \in (\widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}})_e$ *and* $\widetilde{\mu}\{(\eta, \zeta) : \eta \geqslant \zeta \text{ or } \zeta \geqslant \eta\} = 1$ *then*

$$\widetilde{\mu}\{(\eta, \zeta) : \eta \geqslant \zeta\} = 1 \text{ or } \widetilde{\mu}\{(\eta, \zeta) : \zeta \geqslant \eta\} = 1.$$

A crucial lemma needed in the proof of uniqueness is the following.

Lemma 3.12 *Let* $\widetilde{\mu} \in \widetilde{\mathcal{S}}_e$ *such that* $\int [\eta(0) + \zeta(0)] d\widetilde{\mu} < \infty$. *Fix* $x \neq y \in \mathbb{Z}^d$. *Then*

$$\lim_{t \to \infty} \widetilde{\mu}_t \big\{ (\eta, \zeta) : \eta(x) > \zeta(x) \text{ and } \eta(y) < \zeta(y) \big\} = 0$$

Proof Our proof employs some of the notation developed in Sect. 2. Fix a positive integer m. Let $I = [-m, m]^d$ and let B be the event that I contains both $(+)$ and $(-)$ particles. The theorem will be proved if we can show that $\widetilde{\mu}_t(B) \to 0$ as $t \to \infty$. So let us assume to the contrary that we can find a sequence $t_k \uparrow \infty$ such that

$$\widetilde{\mu}_{t_k}(B) \geqslant \delta > 0. \tag{3.11}$$

By our assumptions on the environment, we can find a positive integer $T = T(m)$ and a positive real number $\rho = \rho(m) > 0$ such that

$$\min_{x,\, y \in I} P\big\{ X^x_\bullet \text{ and } \tilde{X}^y_\bullet \text{ meet by time } T \big\} \geqslant \rho.$$

Let $A(t, y)$ denote the event that a $(+)$ or a $(-)$ particle present in the cube $y + I$ at time t has been annihilated by time $t + T$. It is clear that

$$P\big[A(t, y) \big| \eta_t, \zeta_t \big] \geqslant \rho \cdot \mathbb{1}_B \big\{ \theta_y(\eta_t, \zeta_t) \big\} \text{ a.s.} \tag{3.12}$$

For what follows, assume that all $t_{k+1} - t_k \geqslant T$. Let $\phi_t(x) = \beta_t^+(x) + \beta_t^-(x)$ be the number of discrepancy particles at x at time t. Let $n = l(2m + 1) + m$ for a positive integer l and divide the cube $[-n, n]^d$ into $(2l + 1)^d$ cubes of side length $2m + 1$. We have

$$\begin{aligned} \frac{1}{(2n+1)^d} \sum_{y \in [-n,n]^d} \phi_{t_k+T}(y) \leqslant{} & \frac{1}{(2n+1)^d} \sum_{y \in [-n-RT, n+RT]^d} \phi_{t_k}(y) \\ & - \frac{1}{(2n+1)^d} \sum_{j=1}^{(2l+1)^d} \mathbb{1}_B\{\theta_{u(j)}(\eta_{t_k}, \zeta_{t_k})\} \cdot \mathbb{1}_{A(t_k, u(j))} \end{aligned}$$

where $u(j)$ is the center of cube j. Taking expectations and letting $n \to \infty$, we get

$$E_{\widetilde{\mu}}[\phi_{t_k+T}(0)] \leqslant E_{\widetilde{\mu}}[\phi_{t_k}(0)] - \liminf_{n\to\infty} \frac{-1}{(2n+1)^d} \sum_{j=1}^{(2l+1)^d} E_{\widetilde{\mu}}\big[\mathbb{1}_B\{\theta_{u(j)}(\eta_{t_k}, \zeta_{t_k})\} \cdot \mathbb{1}_{A(t_k,u(j))}\big]$$

It follows from (3.12) and (3.11) that

$$E_{\widetilde{\mu}}\big[\mathbb{1}_B\{\theta_{u(j)}(\eta_{t_k}, \zeta_{t_k})\} \cdot \mathbb{1}_{A(t_k,u(j))}\big] \geqslant \rho\widetilde{\mu}_{t_k}(B) \geqslant \rho\delta.$$

We thus have

$$E_{\widetilde{\mu}}[\phi_{t_{k+1}}(0)] \leqslant E_{\widetilde{\mu}}[\phi_{t_k+T}(0)] \leqslant E_{\widetilde{\mu}}[\phi_{t_k}(0)] - \frac{\rho\delta}{(2m+1)^d}.$$

We can conclude from Lemma 2.1 that $E_{\widetilde{\mu}}[\phi_{t_k}(0)] \to -\infty$. But this is a contradiction since $E_{\widetilde{\mu}}[\phi_t(0)] \geqslant 0$. The proof of the lemma is complete. □

Lemma 3.13 *If $\mu_1, \mu_2 \in (\mathcal{I} \cap \mathcal{S})_e$ and $E_{\mu_i}\eta(0) < \infty$ for $i = 1, 2$, then $\mu_1 \leqslant \mu_2$ or $\mu_2 \leqslant \mu_1$.*

Proof From Lemma 3.10, we can find $\widetilde{\mu} \in (\widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}})_e$ with marginals μ_1 and μ_2. Using the ergodic decomposition of stationary measures [24, Theorem 6.6],

$$\begin{aligned}&\widetilde{\mu}\big\{(\eta, \zeta) : \eta(x) > \zeta(x) \text{ and } \eta(y) < \zeta(y)\big\}\\ &\quad = \int_{\widetilde{\mathcal{S}}_e} \tilde{\nu}\big\{(\eta, \zeta) : \eta(x) > \zeta(x) \text{ and } \eta(y) < \zeta(y)\big\}\Psi(d\tilde{\nu}),\end{aligned}$$

for a probability measure Ψ on $\widetilde{\mathcal{S}}_e$. On applying the operator $S(t)$ to both sides of the above equation, we observe that the right hand side goes to 0. We thus get

$$\widetilde{\mu}\{(\eta, \zeta) : \eta \leqslant \zeta \text{ or } \zeta \leqslant \eta\} = 1$$

An application of Lemma 3.11 completes the proof. □

Proposition 3.14 *If $\mu \in (\mathcal{I} \cap \mathcal{S})_e$ and $\rho_0 = E_\mu\eta(0) < \infty$ then $\mu = \mu^{\rho_0}$.*

Proof Since $\mu^\rho \in \mathcal{S}_e \cap \mathcal{I}$, it follows that $\mu^\rho \in (\mathcal{I} \cap \mathcal{S})_e$. We can then conclude from Lemmas 3.9 and 3.13 that there exists a $\rho_0' \in [0, \infty]$ such that $\mu \leqslant \mu^\rho$ for $\rho > \rho_0'$ and $\mu \geqslant \mu^\rho$ for $\rho < \rho_0'$. In particular, we have $\rho_0 = E_\mu\eta(0) \leqslant \rho$ for $\rho > \rho_0'$ and similarly $\rho_0 \geqslant \rho$ for $\rho < \rho_0'$. This says that $\rho_0' = \rho_0$.

Now fix $\rho_1 < \rho_0 < \rho_2$. For all $(x_1, x_2, \cdots, x_n) \in (\mathbb{Z}^d)^n$ and all $(k_1, k_2, \cdots, k_n) \in (\mathbb{Z}_+)^n$, we have

$$\mu^{\rho_1}(\eta(x_i) \geqslant k_i, 1 \leqslant i \leqslant n) \leqslant \mu(\eta(x_i) \geqslant k_i, 1 \leqslant i \leqslant n) \leqslant \mu^{\rho_2}(\eta(x_i) \geqslant k_i, 1 \leqslant i \leqslant n)$$

The first inequality (resp. the second inequality) above can be seen by looking at the coupled measure $\widetilde{\mu}$ corresponding to μ^{ρ_1} (resp. μ^{ρ_2}) and μ so that $\widetilde{\mu}(\eta \leqslant \zeta) = 1$

(resp. $\widetilde{\mu}(\eta \geqslant \zeta) = 1$). Now let $\rho_1 \uparrow \rho_0$ and $\rho_2 \downarrow \rho_0$ to see that μ has the same finite dimensional distributions as μ^{ρ_0}. □

Proof of the remaining parts of Theorem 1.1 We first prove that μ^ρ is the unique measure with the stated properties in part (a) of Theorem 1.1. Indeed, let μ be another measure with those properties. Since $\mu \in \mathcal{S}_e \cap \mathcal{I}$, we can conclude that $\mu \in (\mathcal{I} \cap \mathcal{S})_e$. From Proposition 3.14, we must have that $\mu = \mu^\rho$.

We now turn to part (b) of the theorem. Let ν be a probability measure on $\mathbb{Z}_+^{\mathbb{Z}^d}$ that is stationary and ergodic under spatial translations and has mean occupation $\int \zeta(0)\, d\nu = \rho$. Denote the occupation process with initial distribution ν by ζ_t. Utilizing the ergodic decomposition theorem [24, Theorem 6.6], find $\widetilde{\mu} \in \widetilde{\mathcal{S}}_e$ with marginals μ^ρ and ν. Let $\widetilde{\mu}_t$ be the time t distribution of the joint process (η_t, ζ_t) coupled as described in Sect. 2.

Initial shift invariance implies that mean occupations are constant ρ throughout time and space:

$$E_{\widetilde{\mu}_t}[\zeta(x)] = E_\nu[\zeta_t(x)] = \int \mathbb{E}\Big\{\sum_y \zeta(y)\pi^\omega_{0,t}(y,x)\Big\}\,\nu(d\zeta) = \sum_y \mathbb{E}\big(\pi^\omega_{0,t}(y,x)\big)\int \zeta(y)d\nu = \rho.$$

Chebyshev's inequality and Tychonov's theorem (Theorem 37.3 in [12]) can be used to show that the sequence $\{\widetilde{\mu}_t\}_{t\in\mathbb{Z}_+}$ is tight.

Let $\tilde{\nu}$ be any limit point as $t \to \infty$. Then by Proposition 2.2 $\tilde{\nu}\{(\eta,\zeta) : \eta = \zeta\} = 1$. This proves that $P_\nu\{\zeta_t \in \cdot\} \Rightarrow \mu^\rho$. This completes the proof of Theorem 1.1. □

4 Covariances of the Invariant Measures

Define the Green's functions for both q and $\bar{q}$ walks by

$$G_N(x,y) = \sum_{k=0}^N q^k(x,y) \quad \text{and} \quad \bar{G}_N(x,y) = \sum_{k=0}^N \bar{q}^k(x,y).$$

Recall the potential kernel for the $\bar{q}$ walk

$$\bar{a}(x) = \lim_{N\to\infty}\big(\bar{G}_N(0,0) - \bar{G}_N(x,0)\big). \tag{4.1}$$

In the transient case $d \geqslant 3$ the limit above exists trivially, since

$$G(x,y) = \sum_{k=0}^\infty q^k(x,y) < \infty \quad \text{and} \quad \bar{G}(x,y) = \sum_{k=0}^\infty \bar{q}^k(x,y) < \infty.$$

So for $d \geqslant 3$

$$\bar{a}(x) = \bar{G}(0,0) - \bar{G}(x,0). \tag{4.2}$$

For the existence of the limit (4.1) in the recurrent case $d \in \{1, 2\}$ see T1 on p. 352 of [22]. In all cases the kernel $\bar{a}(x)$ satisfies these equations:

$$\sum_z \bar{q}(0,z)\bar{a}(z) = 1 \quad \text{and} \quad \sum_z \bar{q}(x,z)\bar{a}(z) = \bar{a}(x) \text{ for } x \neq 0. \tag{4.3}$$

The constant β defined by (1.13) has the alternate representation

$$\beta = \sum_z q(0,z)\bar{a}(z). \tag{4.4}$$

We omit the argument for the equality of the two representations of β. It is a simple version of the one given at the end of this section for (1.15).

To prove Theorem 1.2 we first verify this proposition and then derive the Fourier representation (1.15).

Proposition 4.1 *Let $d \geqslant 1$. For $m \in \mathbb{Z}^d \setminus \{0\}$*

$$\mathbb{C}\text{ov}[f(\omega), f(T_{m,0}\omega)] = \beta^{-1} \sum_z q(0,z)[\bar{a}(-m) - \bar{a}(z-m)] \tag{4.5}$$

and

$$\mathbb{V}\text{ar}[f(\omega)] = \beta^{-1} - 1. \tag{4.6}$$

A few more notations. Recall that Y_n denotes the Markov chain with transition q and $\bar{Y}_n$ the $\bar{q}$ random walk. Successive returns to the origin are marked as follows:

$$\tau_0 = 0 \text{ and for } j > 0, \tau_j = \inf\{n > \tau_{j-1} : Y_n = 0\}. \tag{4.7}$$

Abbreviate $\tau = \tau_1$. The corresponding stopping time for $\bar{Y}_n$ is $\bar{\tau}$. For $m \in \mathbb{Z}^d$ and $N \geqslant 1$ abbreviate

$$C_N(m) = \mathbb{C}\text{ov}[f_N(\omega), f_N(T_{m,0}\omega)] = \sum_{z,w \in \mathbb{Z}^d} \mathbb{C}\text{ov}\big[\pi_{-N,0}(z,0),\ \pi_{-N,0}(w,m)\big].$$

Define also the function

$$h(y) = \sum_{z \in \mathbb{Z}^d} \mathbb{C}\text{ov}[\pi_{0,1}(0, y+z),\ \pi_{0,1}(0,z)] = q(0,y) - \bar{q}(0,y), \quad y \in \mathbb{Z}^d.$$

Symmetry $h(-y) = h(y)$ holds.

Lemma 4.2 *In all dimensions $d \geqslant 1$,*

$$C_N(m) = \sum_{y \in \mathbb{Z}^d} h(y) G_{N-1}(y, m). \tag{4.8}$$

Proof The case $N = 1$ follows from a shift of space and time. To do induction on N use the Markov property and the additivity of covariance. Abbreviate temporarily $\kappa_{x,y} = \pi_{-N,-N+1}(x, y)$ and recall that the mean kernel is $p_{y-x} = \mathbb{E}\kappa_{x,y}$.

$$\begin{aligned} C_N(m) = &\sum_{z,z_1,w,w_1} \mathbb{C}\mathrm{ov}\big[\kappa_{z,z_1}\pi_{-N+1,0}(z_1,0)\,,\ \kappa_{w,w_1}\pi_{-N+1,0}(w_1,m)\big] \\ = &\sum_{z,z_1,w,w_1} \Big\{\mathbb{C}\mathrm{ov}\big[(\kappa_{z,z_1} - p_{z_1-z})\pi_{-N+1,0}(z_1,0)\,,\ (\kappa_{w,w_1} - p_{w_1-w})\pi_{-N+1,0}(w_1,m)\big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.9) \\ &+ \mathbb{C}\mathrm{ov}\big[p_{z_1-z}\pi_{-N+1,0}(z_1,0)\,,\ (\kappa_{w,w_1} - p_{w_1-w})\pi_{-N+1,0}(w_1,m)\big] \qquad (4.10) \\ &+ \mathbb{C}\mathrm{ov}\big[p_{z_1-z}\pi_{-N+1,0}(z_1,0)\,,\ p_{w_1-w}\pi_{-N+1,0}(w_1,m)\big]\Big\}. \qquad (4.11) \end{aligned}$$

Working from the bottom up, the terms on line (4.11) add up to $C_{N-1}(m)$. The terms on line (4.10) vanish because $\kappa_{w,w_1} - p_{w_1-w}$ is mean zero and independent of the other random variables inside the covariance. On line (4.9) the covariance vanishes unless $z = w$. Thus by rearranging line (4.9) we get

$$\begin{aligned} &C_N(m) - C_{N-1}(m) = \text{line (4.9)} \\ &= \sum_{z,z_1,w_1} \mathbb{C}\mathrm{ov}(\kappa_{z,z_1}, \kappa_{z,w_1})\mathbb{E}\big[\pi_{-N+1,0}(z_1,0)\pi_{-N+1,0}(w_1,m)\big] \\ &= \sum_{y,x} \mathbb{C}\mathrm{ov}(\kappa_{0,x}, \kappa_{0,x+y}) \sum_{\ell} \mathbb{E}\big[\pi_{-N+1,0}(y, m+\ell)\pi_{-N+1,0}(0,\ell)\big] \\ &= \sum_{y} h(y) q^{N-1}(y,m). \end{aligned}$$

□

In the recurrent case we will use Abel summation, hence the next lemma.

Lemma 4.3 *Let $d \in \{1, 2\}$. For $x, m \in \mathbb{Z}^d$, the limit*

$$a(x,m) = \lim_{s \nearrow 1} \sum_{k=0}^{\infty} s^k \big(q^k(0,m) - q^k(x,m)\big) \tag{4.12}$$

exists. For $m = 0$ the limit is

$$a(x,0) = \frac{\bar{a}(x)}{\beta} \tag{4.13}$$

and for $m \neq 0$

$$a(x,m) = \frac{\bar{a}(x)}{\beta}\sum_z q(0,z)\big[\bar{a}(-m) - \bar{a}(z-m)\big] - \bar{a}(-m) + \bar{a}(x-m). \quad (4.14)$$

Proof Let s vary in (0, 1) and let

$$U(x,m,s) = E_x\Big[\sum_{k=0}^{\tau-1} s^k \mathbf{1}\{Y_k = m\}\Big] \underset{s \nearrow 1}{\longrightarrow} E_x\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big] = U(x,m).$$

Decompose the summation across intervals $[\tau_j, \tau_{j+1})$ and use the Markov property:

$$\begin{aligned}\sum_{k=0}^{\infty} s^k q^k(x,m) &= E_x\Big[\sum_{k=0}^{\tau_1-1} s^k \mathbf{1}\{Y_k = m\}\Big] + \sum_{j=1}^{\infty} E_x\Big[s^{\tau_j} \sum_{k=\tau_j}^{\tau_{j+1}-1} s^{k-\tau_j}\mathbf{1}\{Y_k = m\}\Big] \\ &= U(x,m,s) + \sum_{j=1}^{\infty} E_x(s^\tau)E_0(s^\tau)^{j-1}U(0,m,s) \\ &= U(x,m,s) + \frac{E_x(s^\tau)}{1-E_0(s^\tau)}U(0,m,s).\end{aligned}$$

From this,

$$\sum_{k=0}^{\infty} s^k\big(q^k(0,m) - q^k(x,m)\big) = \frac{1-E_x(s^\tau)}{1-E_0(s^\tau)}U(0,m,s) - U(x,m,s). \quad (4.15)$$

We analyze the quantities on the right in (4.15).

Suppose first $x \neq 0$. Then $U(x,m)$ is the same for the Markov chain Y_k as for the random walk $\bar{Y}_k$ because these processes agree until the first visit to 0. In the notation of Spitzer [22], with a check added to refer to the random walk $\bar{Y}_k$, $\bar{g}_{\{0\}}(x,m) = U(x,m)$. By P29.4 on p. 355 of [22] and D11.1 on p. 115, for recurrent random walk

$$U(x,m) = \bar{g}_{\{0\}}(x,m) = \bar{a}(x) + \bar{a}(-m) - \bar{a}(x-m).$$

For $x = 0$ we have $U(0,0) = 1$, and for $m \neq 0$,

$$\begin{aligned}U(0,m) &= \sum_{y\neq 0} q(0,y)U(y,m) = \sum_{y \neq 0} q(0,y)\big[\bar{a}(y) + \bar{a}(-m) - \bar{a}(y-m)\big] \\ &= \beta + \sum_y q(0,y)\big[\bar{a}(-m) - \bar{a}(y-m)\big].\end{aligned}$$

For the asymptotics of the fraction on the right in (4.15) we can assume again $x \neq 0$ for otherwise the value is 1. It will be convenient to look at the reciprocal. A computation gives

$$\begin{aligned}\frac{1 - E_0(s^\tau)}{1 - E_x(s^\tau)} &= \frac{\sum_{k=0}^\infty s^k P_0(\tau > k)}{\sum_{k=0}^\infty s^k P_x(\tau > k)} \\ &= \frac{1}{\sum_{k=0}^\infty s^k P_x(\tau > k)} \;+\; s \sum_{z \neq 0} q(0,z) \frac{\sum_{k=0}^\infty s^k P_z(\tau > k)}{\sum_{k=0}^\infty s^k P_x(\tau > k)}.\end{aligned}$$

Again we can take advantage of known random walk limits because both $x, z \neq 0$ so the probabilities are the same as those for $\bar{Y}_k$. By P32.2 on p. 379 of [22], as $s \nearrow 1$, for recurrent random walk the above converges to (note that $E_x(\tau) = \infty$)

$$\sum_{z \neq 0} q(0,z) \frac{\bar{a}(z)}{\bar{a}(x)} = \frac{\beta}{\bar{a}(x)}.$$

Letting $s \nearrow 1$ in (4.15) gives (4.13) and (4.14). □

For $m = 0$ we can obtain the convergence as in (4.1) without the Abel summation. But we do not need this for further development.
Proof of Proposition 4.1. Since $f_N \to f$ in $L^2(\mathbb{P})$, the covariance in (4.5) is given by the limit of $C_N(m)$, so by (4.8)

$$\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = \lim_{N \to \infty} \Big\{ \sum_y q(0,y) G_{N-1}(y,m) - \sum_y \bar{q}(0,y) G_{N-1}(y,m) \Big\}.$$

Next,

$$\sum_y q(0,y) G_{N-1}(y,m) = G_N(0,m) - \delta_{0,m} = q^N(0,m) - \delta_{0,m} + G_{N-1}(0,m).$$

Since the Markov chain q follows the random walk $\bar{q}$ away from 0 it is null recurrent for $d = 1, 2$ and transient for $d \geqslant 3$. So $q^N(0,m) \to 0$ [13, Theorem 1.8.5]. Thus the limiting covariance now has the form

$$-\,\delta_{0,m} + \lim_{N \to \infty} \sum_y \bar{q}(0,y)[G_{N-1}(0,m) - G_{N-1}(y,m)]. \tag{4.16}$$

At this point the treatment separates into recurrent and transient cases. This is because the Green's function is uniformly bounded only in the transient case.

Case 1. $d \in \{1, 2\}$
Convergence in (4.16) implies Abel convergence (Theorem 12.41 in [26] or Theorem 1.33 in Chap. 3 of [27]), so the limiting covariance equals

$$-\delta_{0,m} + \lim_{s \nearrow 1} \sum_y \bar{q}(0, y) \sum_{k=0}^{\infty} s^k \big(q^k(0, m) - q^k(y, m)\big).$$

By substituting in (4.13) and (4.14) we obtain (4.6) and (4.5).

Case 2. $d \geqslant 3$
In the transient case we can pass directly to the limit in (4.16) and obtain

$$\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = -\delta_{0,m} + \sum_y \bar{q}(0, y)[G(0, m) - G(y, m)]. \tag{4.17}$$

The sum above can be restricted to $y \neq 0$. By restarting after the first return to 0,

$$G(y, m) = E_y\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big] + P_y(\tau < \infty)G(0, m). \tag{4.18}$$

Next,

$$\begin{aligned} G(0, m) &= \sum_{j=0}^{\infty} E_0\Big[\mathbf{1}\{\tau_j < \infty\} \sum_{k=\tau_j}^{\tau_{j+1}-1} \mathbf{1}\{Y_k = m\}\Big] \\ &= \sum_{j=0}^{\infty} P_0(\tau < \infty)^j E_0\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big] \\ &= \frac{1}{P_0(\tau = \infty)}\Big(\delta_{0,m} + (1 - \delta_{0,m}) \sum_{z \neq 0} q(0, z) E_z\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big]\Big). \end{aligned} \tag{4.19}$$

Now consider first $m \neq 0$. Combining the above,

$$\begin{aligned} &\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = \sum_{y \neq 0} \bar{q}(0, y)\big\{G(0, m) - G(y, m)\big\} \\ &= \sum_{y \neq 0} \bar{q}(0, y)\Big\{ \frac{P_y(\tau = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) E_z\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big] - E_y\Big[\sum_{k=0}^{\tau-1} \mathbf{1}\{Y_k = m\}\Big]\Big\} \end{aligned}$$

using equality of q and $\bar{q}$ away from 0

$$= \sum_{y \neq 0} \bar{q}(0, y) \Big\{ \frac{P_y(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) E_z \Big[\sum_{k=0}^{\bar{\tau}-1} \mathbf{1}\{\bar{Y}_k = m\} \Big] - E_y \Big[\sum_{k=0}^{\bar{\tau}-1} \mathbf{1}\{\bar{Y}_k = m\} \Big] \Big\}$$

$$= \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) E_z \Big[\sum_{k=0}^{\bar{\tau}-1} \mathbf{1}\{\bar{Y}_k = m\} \Big] - \sum_{y \neq 0} \bar{q}(0, y) E_y \Big[\sum_{k=0}^{\bar{\tau}-1} \mathbf{1}\{\bar{Y}_k = m\} \Big]$$

applying (4.18) and (4.19) to the $\bar{q}$ walk

$$= \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) \Big\{ \bar{G}(z, m) - P_z(\bar{\tau} < \infty) \bar{G}(0, m) \Big\} - P_0(\bar{\tau} = \infty) \bar{G}(0, m)$$

$$= \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) \Big\{ \bar{G}(z, m) - P_z(\bar{\tau} < \infty) \bar{G}(0, m) \Big\}$$

$$- \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) P_z(\tau = \infty) \bar{G}(0, m)$$

$$= \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} \sum_{z \neq 0} q(0, z) \big[\bar{G}(z, m) - \bar{G}(0, m) \big].$$

To finish this case, note that

$$\beta = \sum_{z} q(0, z) \bar{a}(z) = \sum_{z \neq 0} q(0, z) (\bar{G}(0, 0) - \bar{G}(z, 0)) = \sum_{z \neq 0} q(0, z) \frac{P_z(\bar{\tau} = \infty)}{P_0(\bar{\tau} = \infty)}$$

$$= \frac{P_0(\tau = \infty)}{P_0(\bar{\tau} = \infty)}.$$

We have arrived at

$$\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] = \beta^{-1} \sum_{z \neq 0} q(0, z) \big[\bar{a}(-m) - \bar{a}(z - m) \big].$$

Return to (4.17)–(4.19) to cover the case $m = 0$:

$$\mathbb{C}\mathrm{ov}[f(\omega), f(\omega)] = \sum_{y} \bar{q}(0, y) [G(0, 0) - G(y, 0)] - 1 = \sum_{y \neq 0} \bar{q}(0, y) \frac{P_y(\tau = \infty)}{P_0(\tau = \infty)} - 1$$

$$= \frac{P_0(\bar{\tau} = \infty)}{P_0(\tau = \infty)} - 1 = \beta^{-1} - 1.$$

This completes the proof of Proposition 4.1. □

Completion of the proof of Theorem 1.2 It remains to prove the Fourier representation (1.15) from (4.5). In several stages symmetry of $\bar{a}$ and the transitions is used.

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}[f(\omega), f(T_{m,0}\omega)] &= \beta^{-1}\sum_z q(0,z)[\bar{a}(m) - \bar{a}(m-z)] \\
&= \lim_{N\to\infty} \beta^{-1}\sum_{k=0}^{N}\sum_z q(0,z)[\bar{q}^k(m-z,0) - \bar{q}^k(m,0)] \\
&= \lim_{N\to\infty} \frac{\beta^{-1}}{(2\pi)^d}\sum_{k=0}^{N}\sum_z q(0,z)\int_{\mathbb{T}^d}[e^{-i\theta\cdot(m-z)} - e^{-i\theta\cdot m}]\bar{\lambda}^k(\theta)\,d\theta \\
&= \lim_{N\to\infty} \frac{-\beta^{-1}}{(2\pi)^d}\int_{\mathbb{T}^d}\cos(\theta\cdot m)\frac{1-\lambda(\theta)}{1-\bar{\lambda}(\theta)}(1-\bar{\lambda}^{N+1}(\theta))\,d\theta \\
&= \frac{-\beta^{-1}}{(2\pi)^d}\int_{\mathbb{T}^d}\cos(\theta\cdot m)\frac{1-\lambda(\theta)}{1-\bar{\lambda}(\theta)}\,d\theta.
\end{aligned}
$$

The last equality comes from $0 \leqslant \bar{\lambda}(\theta) < 1$ for $\theta \in \mathbb{T}^d \setminus \{0\}$ and dominated convergence. The ratio $(1-\lambda(\theta))/(1-\bar{\lambda}(\theta))$ stays bounded as $\theta \to 0$ because both transitions q and $\bar{q}$ have zero mean and $\bar{q}$ has a nonsingular covariance matrix [22, P7 p. 74]. □

5 Convergence of Centered Current Fluctuations

We prove Theorem 1.4 by proving the following proposition. Recall the definition of the current $Y_n(t,r)$ from (1.18), and let $\{Z(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mean zero Gaussian process defined by (1.22) or equivalently through the covariance (1.26). Recall also the definitions

$$\overline{Y}_n(t,r) = n^{-1/4}\big\{Y_n(t,r) - E^\omega[Y_n(t,r)]\big\},$$

$$\overline{Y}_n(\boldsymbol{\theta}) = \sum_{i=1}^{N}\theta_i\overline{Y}_n(t_i,r_i) \quad \text{and} \quad Z(\boldsymbol{\theta}) = \sum_{i=1}^{N}\theta_i Z(t_i,r_i).$$

Proposition 5.1

$$E^\omega\big[\exp\big\{i\overline{Y}_n(\boldsymbol{\theta})\big\}\big] \to \mathbf{E}\big[\exp\big\{iZ(\boldsymbol{\theta})\big\}\big] \textit{ in } \mathbb{P}\textit{-probability.} \tag{5.1}$$

The remainder of the section proves this proposition and thereby Theorem 1.4. We write $\overline{Y}_n(\boldsymbol{\theta})$ as a sum of independent mean zero random variables (under P^ω) so that we can apply Lindeberg-Feller [6]:

$$\overline{Y}_n(\boldsymbol{\theta}) = n^{-1/4}\sum_{i=1}^{N}\theta_i\big\{Y_n(t_i,r_i) - E^\omega Y_n(t_i,r_i)\big\} = W_n = \sum_{m=-\infty}^{\infty}\bar{U}_m \tag{5.2}$$

with

$$\bar{U}_m = \sum_{i=1}^{N} \theta_i \Big(U_m(t_i, r_i)\, \mathbf{1}\{m > 0\} - V_m(t_i, r_i)\, \mathbf{1}\{m \leqslant 0\} \Big), \tag{5.3}$$

and

$$U_m(t,r) = n^{-1/4} \sum_{j=1}^{\eta_0(m)} \mathbf{1}\{X^{m,j}_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor\} \\ - n^{-1/4} E^{\omega}(\eta_0(m))\, P^{\omega}(X^{m}_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor), \tag{5.4}$$

$$V_m(t,r) = n^{-1/4} \sum_{j=1}^{\eta_0(m)} \mathbf{1}\{X^{m,j}_{\lfloor nt \rfloor} > \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor\} \\ - n^{-1/4} E^{\omega}(\eta_0(m))\, P^{\omega}(X^{m}_{\lfloor nt \rfloor} > \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor).$$

The n-dependence is suppressed from the notations $\bar{U}_m$, $U_m(t,r)$ and $V_m(t,r)$. The variables $\{\bar{U}_m\}_{m\in\mathbb{Z}}$ are independent under P^{ω} because initial occupation variables and walks are independent. We will also use repeatedly this formula, a consequence of the independence of η_0 and the walks under P^{ω}:

$$E^{\omega}[U_m(t,r)^2] = n^{-1/2}\, \mathrm{Var}^{\omega}\Big(\sum_{j=1}^{\eta_0(m)} \mathbf{1}\{X^{m,j}_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor \}\Big) \\ = n^{-1/2} E^{\omega}(\eta_0(m))\, P^{\omega}(X^{m}_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor)\, P^{\omega}(X^{m}_{\lfloor nt \rfloor} > \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor) \\ + n^{-1/2}\, \mathrm{Var}^{\omega}(\eta_0(m))\, P^{\omega}(X^{m}_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor)^2 \tag{5.5}$$

and the corresponding formula for $V_m(t,r)$.

Let $a(n) \nearrow \infty$ be a sequence that will be determined precisely in the proof. Define the finite sum

$$W_n^* = \sum_{|m| \leqslant a(n)\sqrt{n}} \bar{U}_m. \tag{5.6}$$

We observe that the terms $|m| > a(n)\sqrt{n}$ can be discarded from (5.2).

Lemma 5.2 $E|W_n - W_n^*|^2 \to 0$ *as* $n \to \infty$.

Proof By the mutual independence of occupation variables and walks under P^{ω}, and as eventually $a(n) > |r_i|$, the task boils down to showing that sums of this type vanish:

$$
\begin{aligned}
E\Big[\Big(\sum_{m>a(n)\sqrt{n}} U_m(t,r)\Big)^2\Big] &= \mathbb{E}\sum_{m>a(n)\sqrt{n}} E^\omega[U_m(t,r)^2] \\
&\leqslant n^{-1/2}\,\mathbb{E}\sum_{m>a(n)\sqrt{n}} \big[E^\omega(\eta_0(m)) + \mathrm{Var}^\omega(\eta_0(m))\big] P^\omega\{X^{m,j}_{\lfloor nt\rfloor} \leqslant \lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor\} \\
&\leqslant Cn^{-1/2}\sum_{m>a(n)\sqrt{n}} P\{X_{\lfloor nt\rfloor} \leqslant \lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor - m\} \\
&= CE\Big[\Big(\frac{X_{\lfloor nt\rfloor} - \lfloor nvt\rfloor}{\sqrt{n}} - r + a(n)\Big)^-\Big].
\end{aligned}
$$

Under the averaged measure P the walk X_s is a sum of bounded i.i.d. random variables, hence by uniform integrability the last line vanishes as $a(n) \nearrow \infty$. There is also a term for $m < a(n)\sqrt{n}$ involving $V_m(t,r)$ that is handled in the same way.

□

The limit $\boldsymbol{\theta}\cdot\mathbf{Z}$ in our goal (5.1) has variance

$$
\sigma_\theta^2 = \sum_{1\leqslant i,j\leqslant N} \theta_i\theta_j\Big[\rho_0\Gamma_1\big((t_i,r_i),(t_j,r_j)\big) + \sigma_0^2\Gamma_2\big((t_i,r_i),(t_j,r_j)\big)\Big] \tag{5.7}
$$

and the two Γ-terms, defined earlier in (1.24) and (1.25), have the following expressions in terms of a standard 1-dimensional Brownian motion B_t:

$$
\begin{aligned}
\Gamma_1\big((s,q),(t,r)\big) = \int_{-\infty}^{\infty}\Big(&\mathbf{P}[B_{\sigma^2 s} \leqslant q-x]\mathbf{P}[B_{\sigma^2 t} > r-x] \\
&- \mathbf{P}[B_{\sigma^2 s} \leqslant q-x,\, B_{\sigma^2 t} > r-x]\Big)\,dx
\end{aligned} \tag{5.8}
$$

and

$$
\begin{aligned}
\Gamma_2\big((s,q),(t,r)\big) = &\int_0^{\infty} \mathbf{P}[B_{\sigma^2 s} \leqslant q-x]\mathbf{P}[B_{\sigma^2 t} \leqslant r-x]\,dx \\
&+ \int_{-\infty}^{0} \mathbf{P}[B_{\sigma^2 s} > q-x]\mathbf{P}[B_{\sigma^2 t} > r-x]\,dx.
\end{aligned} \tag{5.9}
$$

By Lemma 5.2, the desired limit (5.1) follows from showing

$$
E^\omega(e^{iW_n^*}) \to e^{-\sigma_\theta^2/2} \quad \text{in } \mathbb{P}\text{-probability as } n\to\infty. \tag{5.10}
$$

This limit will be achieved by showing that the usual conditions of the Lindeberg-Feller theorem hold in $\mathbb{P}$-probability:

$$
\sum_{|m|\leqslant a(n)\sqrt{n}} E^\omega(\bar{U}_m^2) \to \sigma_\theta^2 \tag{5.11}
$$

and

$$\sum_{|m|\leqslant a(n)\sqrt{n}} E^\omega\big(\,|\bar{U}_m|^2\mathbf{1}\{|\bar{U}_m|\geqslant\varepsilon\}\big)\to 0. \tag{5.12}$$

The standard Lindeberg-Feller theorem can then be applied to subsequences. The limits (5.11)–(5.12) in $\mathbb{P}$-probability imply that every subsequence has a further subsequence along which these limits hold for $\mathbb{P}$-almost every ω. Thus along this further subsequence W_n^* converges weakly to $\mathcal{N}(0,\sigma_\theta^2)$ under P^ω for $\mathbb{P}$-almost every ω. So, every subsequence has a further subsequence along which the limit (5.10) holds for $\mathbb{P}$-almost every ω. This implies the limit (5.10) in $\mathbb{P}$-probability.

We check the negligibility condition (5.12) in the L^1 sense.

Lemma 5.3 *Under assumption* (1.20),

$$\lim_{n\to\infty}\sum_{|m|\leqslant a(n)\sqrt{n}} E\Big[|\bar{U}_m|^2\mathbf{1}\{|\bar{U}_m|\geqslant\varepsilon\}\Big]=0. \tag{5.13}$$

Proof First

$$\begin{aligned}\bar{U}_m^2&=\bigg(\sum_{i=1}^N\theta_i\Big[U_m(t_i,r_i)\mathbf{1}\{m\geqslant 0\}-V_m(t_i,r_i)\mathbf{1}\{m<0\}\Big]\bigg)^2\\&\leqslant C\sum_{i=1}^N U_m(t_i,r_i)^2\mathbf{1}\{m\geqslant 0\}+C\sum_{i=1}^N V_m(t_i,r_i)^2\mathbf{1}\{m<0\}.\end{aligned}$$

The arguments for the terms above are the same. So take a term from the first sum, let $(t,r)=(t_i,r_i)$, and the task is now

$$\lim_{n\to\infty}\sum_{m=0}^{a(n)\sqrt{n}} E\big[U_m(t,r)^2\mathbf{1}\{|\bar{U}_m|\geqslant\varepsilon\}\big]=0. \tag{5.14}$$

Since

$$|\bar{U}_m|\leqslant Cn^{-1/4}\big[\eta_0(m)+E^\omega(\eta(m))\big]$$

and by adjusting ε, limit (5.14) follows if we can show the limit for these sums:

$$\begin{aligned}&\sum_{m=0}^{a(n)\sqrt{n}} E\big[U_m(t,r)^2\mathbf{1}\{\eta_0(m)>n^{1/4}\varepsilon\}\big]\\&\quad+\sum_{m=0}^{a(n)\sqrt{n}} E\big[U_m(t,r)^2\mathbf{1}\{E^\omega(\eta_0(m))>n^{1/4}\varepsilon\}\big].\end{aligned} \tag{5.15}$$

Abbreviate

$$A_m = \{X^m_{nt} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor \}.$$

The terms of the second sum in (5.15) develop as follows, using (5.5), the independence of $\bar{\omega}_{-\infty,-1}$ and $\bar{\omega}_{0,\infty}$, and the shift invariance:

$$\begin{aligned}
&\mathbb{E}\big[E^\omega(U_m(t,r)^2)\mathbf{1}\{E^\omega(\eta_0(m)) > n^{1/4}\varepsilon\}\big] \\
&\leqslant n^{-1/2}\mathbb{E}\big[\big(E^\omega(\eta_0(m)) + \mathrm{Var}^\omega(\eta_0(m))\big)\mathbf{1}\{E^\omega(\eta_0(m)) > n^{1/4}\varepsilon\}\big]P(A_m) \\
&= n^{-1/2}\mathbb{E}\big[\big(E^\omega(\eta_0(0)) + \mathrm{Var}^\omega(\eta_0(0))\big)\mathbf{1}\{E^\omega(\eta_0(0)) > n^{1/4}\varepsilon\}\big]P(A_m).
\end{aligned}$$

Since the averaged walk is a walk with bounded i.i.d. steps,

$$\sum_{m=0}^{a(n)\sqrt{n}} P(A_m) \leqslant E\big[(X_{\lfloor nt \rfloor} - \lfloor nvt \rfloor - \lfloor r\sqrt{n} \rfloor)^-\big] \leqslant C(n^{1/2} + 1). \tag{5.16}$$

Thus

$$\begin{aligned}
&\sum_{m=0}^{a(n)\sqrt{n}} E\big[U_m(t,r)^2\mathbf{1}\{E^\omega(\eta_0(m)) > n^{1/4}\varepsilon\}\big] \\
&\qquad \leqslant C\mathbb{E}\big[\big(E^\omega(\eta_0(0)) + \mathrm{Var}^\omega(\eta_0(0))\big)\mathbf{1}\{E^\omega(\eta_0(0)) > n^{1/4}\varepsilon\}\big].
\end{aligned}$$

The last line vanishes as $n \to \infty$ by dominated convergence, by assumption (1.20).

For the first sum in (5.15) first take quenched expectation of the walks while conditioning on η_0, to get the bound

$$E^\omega_{\eta_0}[U_m(t,r)^2] \leqslant 2n^{-1/2}P^\omega(A_m)\big[\eta_0(m)^2 + E^\omega(\eta_0(m))^2\big].$$

Using again the independence of $\bar{\omega}_{-\infty,-1}$ and $\bar{\omega}_{0,\infty}$, shift-invariance, and (5.16),

$$\begin{aligned}
&\sum_{m=0}^{a(n)\sqrt{n}} E\big[U_m(t,r)^2\mathbf{1}\{\eta_0(m) > n^{1/4}\varepsilon\}\big] \\
&\leqslant Cn^{-1/2}\sum_{m=0}^{a(n)\sqrt{n}} P(A_m)\cdot E\big[\big(\eta_0(0)^2 + E^\omega(\eta_0(0))^2\big)\mathbf{1}\{\eta_0(0) > n^{1/4}\varepsilon\}\big] \\
&\leqslant CE\big[\big(\eta_0(0)^2 + E^\omega(\eta_0(0))^2\big)\mathbf{1}\{\eta_0(0) > n^{1/4}\varepsilon\}\big]
\end{aligned}$$

The last line vanishes as $n \to \infty$ by dominated convergence, by assumption (1.20).

□

We turn to checking (5.11).

$$\sum_{|m|\leqslant a(n)\sqrt{n}} E^{\omega}\big[\bar{U}_m^2\big] = \sum_{1\leqslant i,j\leqslant N} \theta_i\theta_j \sum_{|m|\leqslant a(n)\sqrt{n}} \Big[\mathbf{1}_{\{m>0\}} E^{\omega}\big(U_m(t_i,r_i)U_m(t_j,r_j)\big) + \mathbf{1}_{\{m\leqslant 0\}} E^{\omega}\big(V_m(t_i,r_i)V_m(t_j,r_j)\big)\Big].$$

Each quenched expectation of a product of two mean zero random variables is handled in the manner of (5.5) that we demonstrate with the second expectation:

$$\begin{aligned}
&E^{\omega}\big(V_m(t_i,r_i)V_m(t_j,r_j)\big)\\
&= n^{-1/2}\,\mathrm{Cov}^{\omega}\Bigg(\sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X^{m,k}_{\lfloor nt_i\rfloor} > \lfloor nvt_i\rfloor + r_i\sqrt{n}\},\ \sum_{\ell=1}^{\eta_0(m)} \mathbf{1}\{X^{m,\ell}_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + r_j\sqrt{n}\}\Bigg)\\
&= n^{-1/2} E^{\omega}(\eta_0(m))\Big[P^{\omega}(X^m_{\lfloor nt_i\rfloor} > \lfloor nvt_i\rfloor + r_i\sqrt{n},\ X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + r_j\sqrt{n})\\
&\quad - P^{\omega}(X^m_{\lfloor nt_i\rfloor} > \lfloor nvt_i\rfloor + r_i\sqrt{n})P^{\omega}(X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + r_j\sqrt{n})\Big]\\
&\quad + n^{-1/2}\,\mathrm{Var}^{\omega}(\eta_0(m))P^{\omega}(X^m_{\lfloor nt_i\rfloor} > \lfloor nvt_i\rfloor + r_i\sqrt{n})P^{\omega}(X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + r_j\sqrt{n}).
\end{aligned}$$

After some rearranging of the resulting probabilities, we arrive at

$$\begin{aligned}
&\sum_{|m|\leqslant a(n)\sqrt{n}} E^{\omega}\big[\bar{U}_m^2\big]\\
&= n^{-1/2}\sum_{1\leqslant i,j\leqslant N}\theta_i\theta_j\Bigg[\sum_{|m|\leqslant a(n)\sqrt{n}} E^{\omega}(\eta_0(m))\\
&\quad\times\Big\{P^{\omega}(X^m_{\lfloor nt_i\rfloor}\leqslant \lfloor nvt_i\rfloor + \lfloor r_i\sqrt{n}\rfloor)P^{\omega}(X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + \lfloor r_j\sqrt{n}\rfloor)\\
&\qquad - P^{\omega}(X^m_{\lfloor nt_i\rfloor}\leqslant \lfloor nvt_i\rfloor + \lfloor r_i\sqrt{n}\rfloor,\ X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + \lfloor r_j\sqrt{n}\rfloor)\Big\}\\
&+ \sum_{|m|\leqslant a(n)\sqrt{n}}\mathrm{Var}^{\omega}(\eta_0(m))\\
&\quad\times\Big\{\mathbf{1}_{\{m>0\}}P^{\omega}(X^m_{\lfloor nt_i\rfloor}\leqslant \lfloor nvt_i\rfloor + \lfloor r_i\sqrt{n}\rfloor)P^{\omega}(X^m_{\lfloor nt_j\rfloor}\leqslant \lfloor nvt_j\rfloor + \lfloor r_j\sqrt{n}\rfloor)\\
&\quad + \mathbf{1}_{\{m\leqslant 0\}}P^{\omega}(X^m_{\lfloor nt_i\rfloor} > \lfloor nvt_i\rfloor + \lfloor r_i\sqrt{n}\rfloor)P^{\omega}(X^m_{\lfloor nt_j\rfloor} > \lfloor nvt_j\rfloor + \lfloor r_j\sqrt{n}\rfloor)\Big\}\Bigg].
\end{aligned} \tag{5.17}$$

The terms above have been arranged so that the sums match up with the integrals in (5.7)–(5.9). Limit (5.11) is now proved by showing that, term by term, the sums above converge to the integrals. In each case the argument is the same. We illustrate the case of the sum of the first term with the factor $\mathrm{Var}^{\omega}(\eta_0(m))$ in front. To simplify notation we let $((s,q),(t,r)) = ((t_i,r_i),(t_j,r_j))$. In other words, we show this convergence in $\mathbb{P}$-probability:

$$S_0(n) \equiv n^{-1/2} \sum_{0<m\leqslant a(n)\sqrt{n}} \mathrm{Var}^{\omega}(\eta_0(m)) P^{\omega}(X^m_{\lfloor ns \rfloor} \leqslant \lfloor nvs \rfloor + \lfloor q\sqrt{n} \rfloor) \times P^{\omega}(X^m_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor) \tag{5.18}$$
$$\underset{n\to\infty}{\longrightarrow} \sigma_0^2 \int_0^{\infty} \mathbf{P}[B_{\sigma^2 s} \leqslant q - x]\mathbf{P}[B_{\sigma^2 t} \leqslant r - x]\, dx \equiv I.$$

The proof of $S_0(n) \overset{\mathbb{P}}{\to} I$ is divided into two lemmas. Let

$$S_1(n) = n^{-1/2} \sum_{0<m\leqslant a(n)\sqrt{n}} \mathrm{Var}^{\omega}(\eta_0(m)) \times \mathbf{P}\Big(B_{\sigma^2 s} \leqslant q - \frac{m}{\sqrt{n}}\Big)\mathbf{P}\Big(B_{\sigma^2 t} \leqslant r - \frac{m}{\sqrt{n}}\Big). \tag{5.19}$$

Lemma 5.4 $\lim_{n\to\infty} \mathbb{E}|S_0(n) - S_1(n)| = 0.$

Proof By the quenched central limit theorem for space-time RWRE [16], for each $x \in \mathbb{R}$ the limit

$$P^{\omega}(X_{\lfloor ns \rfloor} \leqslant \lfloor nvs \rfloor + \lfloor x\sqrt{n} \rfloor) \to \mathbf{P}(B_{\sigma^2 s} \leqslant x)$$

holds for $\mathbb{P}$-a.e. ω. Since these are distribution functions (monotone and between 0 and 1) with a continuous limit the convergence is uniform in x. Set

$$D_n(\omega) = \sup_{x,y\in\mathbb{R}} \Big| P^{\omega}(X_{\lfloor ns \rfloor} \leqslant \lfloor nvs \rfloor + \lfloor x\sqrt{n} \rfloor) P^{\omega}(X_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor y\sqrt{n} \rfloor) - \mathbf{P}(B_{\sigma^2 s} \leqslant x)\mathbf{P}(B_{\sigma^2 t} \leqslant y) \Big|$$

and then $D_n(\omega) \to 0$ $\mathbb{P}$-a.s. By shift-invariance

$$\begin{aligned}
\mathbb{E}|S_0(n) - S_1(n)| &\leqslant n^{-1/2} \sum_{0<m\leqslant a(n)\sqrt{n}} \mathbb{E}\, \mathrm{Var}^{T_{m,0}\omega}(\eta_0(0)) \\
&\quad \times \Big| P^{T_{m,0}\omega}(X_{\lfloor ns \rfloor} \leqslant \lfloor nvs \rfloor + \lfloor q\sqrt{n} \rfloor - m) P^{T_{m,0}\omega}(X_{\lfloor nt \rfloor} \leqslant \lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor - m) \\
&\qquad - \mathbf{P}\Big(B_{\sigma^2 s} \leqslant q - \frac{m}{\sqrt{n}}\Big)\mathbf{P}\Big(B_{\sigma^2 t} \leqslant r - \frac{m}{\sqrt{n}}\Big) \Big| \\
&\leqslant n^{-1/2} \sum_{0<m\leqslant a(n)\sqrt{n}} \mathbb{E}\Big[\mathrm{Var}^{T_{m,0}\omega}(\eta_0(0)) D_n(T_{m,0}\omega)\Big] \\
&\leqslant 2a(n)\mathbb{E}\Big[\mathrm{Var}^{\omega}(\eta_0(0)) D_n(\omega)\Big].
\end{aligned} \tag{5.20}$$

Moment assumption (1.20) and dominated convergence guarantee that

$$\mathbb{E}\Big[\mathrm{Var}^{\omega}(\eta_0(0)) D_n(\omega)\Big] \longrightarrow 0.$$

Thus we can take

$$a(n) = \Big(\sup_{k:k \geqslant n} \mathbb{E}\big[\mathrm{Var}^{\omega}(\eta_0(0)) D_k(\omega)\big] \Big)^{-1/2} \tag{5.21}$$

to have $a(n) \nearrow \infty$ while still line (5.20) vanishes as $n \to \infty$. □

The choice of $a(n)$ made above depends on s, t but that is not problematic since we have only finitely many time points t_i to handle.

Lemma 5.5 $\lim_{n\to\infty} \mathbb{E}|S_1(n) - I| = 0.$

Proof First we discard tails of the sum and integral. Given $\varepsilon > 0$, we can choose a large enough $c < \infty$ such that

$$S_1^*(n) = n^{-1/2} \sum_{0<m\leqslant c\sqrt{n}} \mathrm{Var}^{\omega}(\eta_0(m)) \times \mathbf{P}\Big(B_{\sigma^2 s} \leqslant q - \frac{m}{\sqrt{n}}\Big)\mathbf{P}\Big(B_{\sigma^2 t} \leqslant r - \frac{m}{\sqrt{n}}\Big)$$

satisfies $\mathbb{E}|S_1(n) - S_1^*(n)| \leqslant \varepsilon$, and so that

$$I^* = \sigma_0^2 \int_0^c \mathbf{P}[B_{\sigma^2 s} \leqslant q - x]\mathbf{P}[B_{\sigma^2 t} \leqslant r - x]\, dx$$

satisfies $I - I^* \leqslant \varepsilon$. Thus it suffices to prove $S_1^*(n) \to I^*$.

Next, since the Gaussian distribution functions are Lipschitz continuous,

$$S_1^*(n) - I^* = n^{-1/2} \sum_{0<m\leqslant c\sqrt{n}} \big[\mathrm{Var}^{\omega}(\eta_0(m)) - \sigma_0^2\big] \times \mathbf{P}\Big(B_{\sigma^2 s} \leqslant q - \frac{m}{\sqrt{n}}\Big)\mathbf{P}\Big(B_{\sigma^2 t} \leqslant r - \frac{m}{\sqrt{n}}\Big) + O(n^{-1/2}).$$

Introduce an intermediate scale $1 << L << \sqrt{n}$ and use again Lipschitz continuity of the probabilities:

$$S_1^*(n) - I^* = \frac{L}{n^{1/2}} \sum_{0\leqslant j\leqslant \frac{c\sqrt{n}}{L}-1} \Big(\frac{1}{L} \sum_{m=jL+1}^{(j+1)L} \mathrm{Var}^{\omega}(\eta_0(m)) - \sigma_0^2 \Big) \times \Big\{ \mathbf{P}\Big(B_{\sigma^2 s} \leqslant q - \frac{jL}{\sqrt{n}}\Big)\mathbf{P}\Big(B_{\sigma^2 t} \leqslant r - \frac{jL}{\sqrt{n}}\Big) + O\Big(\frac{L}{\sqrt{n}}\Big) \Big\} + \frac{R_n}{\sqrt{n}} + O(n^{-1/2}).$$

The error term R_n consists of order L terms bounded by $|\mathrm{Var}^{\omega}(\eta_0(m)) - \sigma_0^2|$ that appear because the collection of summation intervals $(jL, (j+1)L]$ may not exactly cover the original summation interval $0 < m \leqslant c\sqrt{n}$. It satisfies $\mathbb{E}R_n \leqslant CL$. Finally, bounding the probabilities crudely by 1 and by shift-invariance,

$$\mathbb{E}|S_1^*(n) - I^*| \leqslant C\mathbb{E}\left|\frac{1}{L}\sum_{m=1}^{L}\mathrm{Var}^{\omega}(\eta_0(m)) - \sigma_0^2\right| + O(Ln^{-1/2}).$$

This vanishes as we let first $n \to \infty$ and then $L \to \infty$ and apply the L^1 ergodic theorem. □

Limit (5.18) has now been verified. All terms in (5.17) are treated the same way to show that they converge, in $L^1(\mathbb{P})$ and therefore in $\mathbb{P}$-probability, to the corresponding integrals in (5.7)–(5.9). This verifies limit (5.11). Since both (5.11) and (5.12) have been checked, the Gaussian limit in (5.10) has been proved, as explained in the paragraph following (5.12). The proof of Proposition 5.1 and thereby also the proof of Theorem 1.4 are complete. □

6 The Quenched Mean Process

We now prove Theorems 1.6 and 1.7. We will use a simplified notation for the quenched jump probabilities: $\omega_{x,n} = \omega_{x,n}(1)$ and $\omega'_{x,n} = \omega_{x,n}(0) = 1 - \omega_{x,n}(1)$. Note that when the steps are 0 and 1 we have $v = p(1) = \mathbb{E}\omega_{0,0}$. Potential kernel $\bar{a}$ can be easily computed from Eq. (4.3) and seen to equal $\bar{a}(x) = \frac{|x|}{2v(1-v)}$. Recall that $\alpha = \mathbb{E}\omega_{0,0}\omega'_{0,0}$. Then formula (4.4) gives

$$\beta = \frac{\alpha}{v(1-v)}\,.$$

Proof of Theorem 1.6 Define

$$H_n(x) = E^{\omega}\Big[\sum_{y>0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} \leqslant x\} - \sum_{y\leqslant 0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} > x\}\Big].$$

Then $Y_n(t,r) = H_{\lfloor nt\rfloor}(\lfloor nvt\rfloor + \lfloor r\sqrt{n}\rfloor)$. Compute

$$\begin{aligned}H_{n+1}(x) = E^{\omega}\Big[\sum_{y>0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} \leqslant x-1\}\Big] + E^{\omega}\Big[\sum_{y>0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} = x\}\Big]\omega'_{x,n}\\ - \sum_{y\leqslant 0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} > x\}\Big] - \sum_{y\leqslant 0}\sum_{j=1}^{\eta_0(y)}\mathbf{1}\{X_n^{y,j} = x\}\Big]\omega_{x,n}.\end{aligned}$$

Also,

$$\begin{aligned}
&\omega_{x,n} H_n(x-1) + \omega'_{x,n} H_n(x) \\
&\quad = E^\omega \Big[\sum_{y>0} \sum_{j=1}^{\eta_0(y)} \mathbf{1}\{X_n^{y,j} \leqslant x-1\} \Big] \omega_{x,n} - \sum_{y\leqslant 0} \sum_{j=1}^{\eta_0(y)} \mathbf{1}\{X_n^{y,j} > x-1\} \Big] \omega_{x,n} \\
&\qquad + E^\omega \Big[\sum_{y>0} \sum_{j=1}^{\eta_0(y)} \mathbf{1}\{X_n^{y,j} \leqslant x\} \Big] \omega'_{x,n} - \sum_{y\leqslant 0} \sum_{j=1}^{\eta_0(y)} \mathbf{1}\{X_n^{y,j} > x\} \Big] \omega'_{x,n}.
\end{aligned}$$

Taking the difference of the two expressions one finds that

$$H_{n+1}(x) = \omega_{x,n} H_n(x-1) + \omega'_{x,n} H_n(x).$$

In other words, H is the random average process introduced by Ferrari and Fontes [8]. The initial conditions are given by

$$H_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sum_{y=1}^{x} E^\omega \eta_0(y) & \text{if } x > 0, \quad \text{and} \\ \sum_{y=x+1}^{0} E^\omega \eta_0(y) & \text{if } x < 0. \end{cases}$$

The claim now follows by applying [20, Theorem 4.1] and the characterization on p. 13 of [20]. ([20, Theorem 4.1] as reproduced from [2, Theorem 2.1] where the limiting stochastic heat equation is slightly altered because the process studied was $H_{\lfloor nt \rfloor}(\lfloor nvt \rfloor + \lfloor r\sqrt{n} \rfloor) - H_0(\lfloor r\sqrt{n} \rfloor))$. □

Proof of Theorem 1.7 Now, we have $\beta = \alpha/(v(1-v)) = 4\alpha$. We will write $p^k_{x,y}$ for the k-step averaged transition. For $t \geqslant 0$ define

$$Y(t) = \sum_{x>0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X_{\lfloor t \rfloor}^{x,j} \leqslant \lfloor vt \rfloor\} - \sum_{x\leqslant 0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X_{\lfloor t \rfloor}^{x,j} > \lfloor vt \rfloor\}.$$

By stationarity

$$E^\omega Y_n(t,0) - E^\omega Y_n(s,0) = E^\omega Y(nt) - E^\omega Y(ns)$$

has the same distribution as the E^ω-mean of

$$Y' = \sum_{x>0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{x,j} \leqslant \lfloor nvt \rfloor - \lfloor nvs \rfloor\} - \sum_{x\leqslant 0} \sum_{j=1}^{\eta_0(x)} \mathbf{1}\{X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{x,j} > \lfloor nvt \rfloor - \lfloor nvs \rfloor\}.$$

The difference $|Y' - Y(\lfloor nt\rfloor - \lfloor ns\rfloor)|$ is bounded by the number of particles that are at time $\lfloor nt\rfloor - \lfloor ns\rfloor$ between $\lfloor nvt\rfloor - \lfloor nvs\rfloor$ and $\lfloor(\lfloor nt\rfloor - \lfloor ns\rfloor)v\rfloor$. Since $|\lfloor nvt\rfloor - \lfloor nvs\rfloor - \lfloor(\lfloor nt\rfloor - \lfloor ns\rfloor)v\rfloor| \leqslant 2$ we are talking about at most 5 sites and, consequently, $\mathbb{E}[|E^\omega Y' - E^\omega Y(\lfloor nt\rfloor - \lfloor ns\rfloor)|] \leqslant 5\mathbb{E}[f] = 5$. A similar reasoning gives a bound on $\mathbb{E}[|Y(nt) - Y(\lfloor nt\rfloor)|]$ and $\mathbb{E}[|Y(ns) - Y(\lfloor ns\rfloor)|]$. Therefore,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{V}\mathrm{ar}\big(E^\omega Y(\lfloor nt\rfloor - \lfloor ns\rfloor)\big) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{V}\mathrm{ar}\big(E^\omega Y_n(t,0) - E^\omega Y_n(s,0)\big)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \Big[\mathbb{V}\mathrm{ar}\big(E^\omega Y(\lfloor nt\rfloor)\big) + \mathbb{V}\mathrm{ar}\big(E^\omega Y(\lfloor ns\rfloor)\big) - 2\mathbb{C}\mathrm{ov}\big(E^\omega Y_n(s,0), E^\omega Y_n(t,0)\big)\Big].$$

Hence, it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{V}\mathrm{ar}\big(E^\omega Y(n)\big) = \frac{1}{\sqrt{2\pi}}\big(\tfrac{1}{4}\alpha^{-1} - 1\big).$$

Since

$$E^\omega Y(2n+1) - E^\omega Y(2n) = -f(T_{n,2n}\omega)\omega_{n,2n}$$

we see that it is enough to prove the above limit along the subsequence of even integers.

Let

$$h(\omega) = f(T_{1,0}\omega)\omega'_{1,0}\omega'_{1,1} - f(\omega)\omega_{0,0}\omega_{1,1}.$$

Then $\mathbb{E}(h) = 0$, $\mathbb{E}(h^2) = \frac{1}{8\alpha} - \frac{1}{2}$ (here we use Corollary 1.3 and $p_0 = p_1 = 1/2$), and

$$E^\omega Y(2n+2) - E^\omega Y(2n) = h(T_{n,2n}\omega).$$

Let $c_0 = \mathbb{V}\mathrm{ar}(f) = \beta^{-1} - 1$. To compute $\mathbb{E}h(\omega)h(T_{n,2n}\omega)$ write

$$h(\omega) = (f(T_{1,0}\omega) - 1)\omega'_{1,0}\omega'_{1,1} - (f(\omega) - 1)\omega_{0,0}\omega_{1,1} + \omega'_{1,0}\omega'_{1,1} - \omega_{0,0}\omega_{1,1}$$

and

$$\begin{aligned} h(T_{n,2n}\omega) = &\sum_{-n+1\leqslant x\leqslant n+1} (f(T_{x,0}\omega) - 1)\pi_{0,2n}(x, n+1)\omega'_{n+1,2n}\omega'_{n+1,2n+1} \\ &- \sum_{-n\leqslant y\leqslant n} (f(T_{y,0}\omega) - 1)\pi_{0,2n}(y, n)\omega_{n,2n}\omega_{n+1,2n+1} \\ &+ \sum_{-n+1\leqslant x\leqslant n+1} \pi_{0,2n}(x, n+1)\omega'_{n+1,2n}\omega'_{n+1,2n+1} \\ &- \sum_{-n\leqslant y\leqslant n} \pi_{0,2n}(y, n)\omega_{n,2n}\omega_{n+1,2n+1}. \end{aligned}$$

Due to $\mathfrak{S}_{-\infty,-1}$-measurability the f-terms are independent of the ω's. Also, distinct shifts are uncorrelated by Corollary 1.3. Multiplying these terms together and separating the expectations of the factors on levels $2n$ and $2n+1$ leads to

$$\mathbb{E}h(\omega)h(T_{n,2n}\omega) = \tfrac{1}{4}c_0\mathbb{E}\pi_{0,2n}(1,n+1)\omega'_{1,0}\omega'_{1,1} \tag{6.1}$$
$$- \tfrac{1}{4}c_0\mathbb{E}\pi_{0,2n}(0,n+1)\omega_{0,0}\omega_{1,1} \tag{6.2}$$
$$- \tfrac{1}{4}c_0\mathbb{E}\pi_{0,2n}(1,n)\omega'_{1,0}\omega'_{1,1} \tag{6.3}$$
$$+ \tfrac{1}{4}c_0\mathbb{E}\pi_{0,2n}(0,n)\omega_{0,0}\omega_{1,1} \tag{6.4}$$
$$+ \tfrac{1}{4}\sum_{-n+1\leqslant x\leqslant n+1}\mathbb{E}\pi_{0,2n}(x,n+1)\big(\omega'_{1,0}\omega'_{1,1}-\omega_{0,0}\omega_{1,1}\big) \tag{6.5}$$
$$- \tfrac{1}{4}\sum_{-n\leqslant y\leqslant n}\mathbb{E}\pi_{0,2n}(y,n)\big(\omega'_{1,0}\omega'_{1,1}-\omega_{0,0}\omega_{1,1}\big). \tag{6.6}$$

Thinking through the possible jumps shows that the terms in (6.5) and (6.6) survive only for $x, y \in \{0,1\}$. And some of these terms can be combined with the ones above. This gives

$$\mathbb{E}h(\omega)h(T_{n,2n}\omega) = \tfrac{1}{4}(c_0+1)\mathbb{E}\pi_{0,2n}(1,n+1)\omega'_{1,0}\omega'_{1,1} \tag{6.7}$$
$$- \tfrac{1}{4}(c_0+1)\mathbb{E}\pi_{0,2n}(0,n+1)\omega_{0,0}\omega_{1,1} \tag{6.8}$$
$$- \tfrac{1}{4}(c_0+1)\mathbb{E}\pi_{0,2n}(1,n)\omega'_{1,0}\omega'_{1,1} \tag{6.9}$$
$$+ \tfrac{1}{4}(c_0+1)\mathbb{E}\pi_{0,2n}(0,n)\omega_{0,0}\omega_{1,1} \tag{6.10}$$
$$+ \tfrac{1}{4}\mathbb{E}\big[\pi_{0,2n}(0,n+1)\omega'_{1,0}\omega'_{1,1} - \pi_{0,2n}(1,n+1)\omega_{0,0}\omega_{1,1}\big] \tag{6.11}$$
$$- \tfrac{1}{4}\mathbb{E}\big[\pi_{0,2n}(0,n)\omega'_{1,0}\omega'_{1,1} - \pi_{0,2n}(1,n)\omega_{0,0}\omega_{1,1}\big]. \tag{6.12}$$

Now transform each term. For example, term (6.7) becomes

$$\begin{aligned}&(6.7)\\ &= \tfrac{1}{4}(c_0+1)\mathbb{E}\Big[\big(\omega'_{1,0}\omega'_{1,1}p^{2n-2}_{1,n+1} + \omega'_{1,0}\omega_{1,1}p^{2n-2}_{2,n+1} + \omega_{1,0}\omega'_{2,1}p^{2n-2}_{2,n+1} + \omega_{1,0}\omega_{1,1}p^{2n-2}_{3,n+1}\big)\,\omega'_{1,0}\omega'_{1,1}\Big]\\ &= \tfrac{1}{4}(c_0+1)\Big\{(\tfrac{1}{2}-\alpha)^2p^{2n-2}_{1,n+1} + (\tfrac{3}{4}\alpha-\alpha^2)p^{2n-2}_{2,n+1} + \tfrac{1}{4}\alpha p^{2n-2}_{3,n+1}\Big\}.\end{aligned}$$

After these steps we get

$$\begin{aligned}&\mathbb{E}h(\omega)h(T_{n,2n}\omega)\\ &\quad= \tfrac{1}{4}(c_0+1)\Big[-\tfrac{1}{4}\alpha(p^{2n-2}_{0,n+1}+p^{2n-2}_{0,n-3})\\ &\qquad\qquad + (2\alpha^2-\tfrac{3}{2}\alpha+\tfrac{1}{4})(p^{2n-2}_{0,n}+p^{2n-2}_{0,n-2}) + (-4\alpha^2+\tfrac{7}{2}\alpha-\tfrac{1}{2})p^{2n-2}_{0,n-1}\Big]\\ &\qquad\qquad + \tfrac{1}{4}\Big[\tfrac{1}{16}(p^{2n-2}_{0,n+1}+p^{2n-2}_{0,n-3}) + (\tfrac{1}{8}-\tfrac{1}{2}\alpha)(p^{2n-2}_{0,n}+p^{2n-2}_{0,n-2}) + (\alpha-\tfrac{3}{8})p^{2n-2}_{0,n-1}\Big].\end{aligned}$$

Letting X_k denote the (averaged) Markov chain with transition $p_{x,y}$, introduce $Z_k = X_{2k} - k$ with transition $r_{0,0} = 1/2, r_{0,\pm 1} = 1/4$. For higher order transitions $p_{0,n+i}^{2n-2} = r_{i+1}^{n-1}$. Replace the p's with r's and combine them using symmetry: $r_2^{n-1} = r_{-2}^{n-1}$, etc. Then

$$\begin{aligned}\mathbb{E}h(\omega)h(T_{n,2n}\omega) &= \tfrac{1}{4}(c_0+1)\Big[\tfrac{1}{2}\alpha(r_0^{n-1} - r_2^{n-1}) - (4\alpha^2 - 3\alpha + \tfrac{1}{2})(r_0^{n-1} - r_1^{n-1})\Big] \\ &\quad + \tfrac{1}{4}\Big[-\tfrac{1}{8}(r_0^{n-1} - r_2^{n-1}) + (\alpha - \tfrac{1}{4})(r_0^{n-1} - r_1^{n-1})\Big] \\ &= \tfrac{1}{4}(\tfrac{1}{2} - \tfrac{1}{8}\alpha^{-1})(r_0^{n-1} - r_1^{n-1}) \end{aligned} \tag{6.13}$$

where in the last step we used $c_0 + 1 = (4\alpha)^{-1}$.

Use the potential kernel a^Z of the r-walk: the variance is $1/2$ so $a^Z(x) = 2|x|$. From [22] and symmetry, $a^Z(x) = \lim_{m\to\infty} a_m^Z(x)$ with

$$a_m^Z(x) = \sum_{k=0}^{m}(r_{0,0}^k - r_{x,0}^k) = \sum_{k=0}^{m}(r_0^k - r_x^k). \tag{6.14}$$

Then

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big(E^\omega Y(2n)\big) &= \mathbb{V}\mathrm{ar}\Big[\sum_{k=0}^{n-1} h(T_{k,2k}\omega)\Big] \\ &= n\mathbb{E}(h^2) + 2\sum_{k=1}^{n-1}(n-k)\mathbb{E}h(\omega)h(T_{k,2k}\omega) \\ &= \big(\tfrac{1}{16}\alpha^{-1} - \tfrac{1}{4}\big)\Big[2n - \sum_{k=1}^{n-1}(n-k)(r_0^{k-1} - r_1^{k-1})\Big] \\ &= \big(\tfrac{1}{16}\alpha^{-1} - \tfrac{1}{4}\big)\Big[2n - \sum_{j=1}^{n-1}\sum_{k=1}^{j}(r_0^{k-1} - r_1^{k-1})\Big] \\ &= \big(\tfrac{1}{16}\alpha^{-1} - \tfrac{1}{4}\big)\Big[na^Z(1) - \sum_{j=1}^{n-1} a_{j-1}^Z(1)\Big]\Big] \\ &= \big(\tfrac{1}{16}\alpha^{-1} - \tfrac{1}{4}\big)\Big[a^Z(1) + \sum_{j=1}^{n-1}(a^Z(1) - a_{j-1}^Z(1))\Big]. \end{aligned} \tag{6.15}$$

Let us look at $a_k^Z(x)$. The characteristic function is

$$\zeta(\theta) = \sum_x r_x e^{ix\theta} = \tfrac{1}{2}(1 + \cos\theta).$$

By symmetry:

$$\begin{aligned} a_m^Z(x) &= \sum_{k=0}^{m}(r_0^k - r_x^k) = \sum_{k=0}^{m} \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(\zeta^k(\theta) - e^{-ix\theta}\zeta^k(\theta)\big)\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\cos x\theta}{1-\zeta(\theta)} \big(1-\zeta^{m+1}(\theta)\big)\, d\theta \\ &= \frac{2}{2\pi} \int_{-\pi}^{\pi} \frac{1-\cos x\theta}{1-\cos\theta} \big(1-\zeta^{m+1}(\theta)\big)\, d\theta \\ &= 2|x| - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1-\cos x\theta}{1-\cos\theta} \left(\frac{1+\cos\theta}{2}\right)^{m+1} d\theta. \end{aligned}$$

The value of the first integral above is on p. 61 in [22]. But actually we only need $x = 1$:

$$\begin{aligned} a_m^Z(1) &= a^Z(1) - \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1+\cos\theta}{2}\right)^{m+1} d\theta = a^Z(1) - \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2m+2} x \, dx \\ &= a^Z(1) - 2 \prod_{\ell=1}^{m+1} \left(1 - \frac{1}{2\ell}\right) = a^Z(1) - \frac{2}{(m+1)!} \prod_{\ell=1}^{m+1} (\ell - \tfrac{1}{2}). \end{aligned}$$

Put this back into (6.15):

$$\begin{aligned} \mathbb{V}\mathrm{ar}\big(E^\omega Y(2n)\big) &= \big(\tfrac{1}{16}\alpha^{-1} - \tfrac{1}{4}\big)\left[a^Z(1) + \sum_{j=1}^{n-1} \frac{2}{j!} \prod_{\ell=1}^{j} (\ell - \tfrac{1}{2})\right] \\ &= \big(\tfrac{1}{8}\alpha^{-1} - \tfrac{1}{2}\big)\left[1 + \sum_{j=1}^{n-1} \frac{j^{-1/2}}{\Gamma_j(-1/2)\cdot(-1/2)}\right]. \end{aligned}$$

Above we used the definition

$$\Gamma_m(x) = \frac{m!\, m^x}{x(x+1)\cdots(x+m)}.$$

According to p. 461 of [23], $\Gamma_m(x) \to \Gamma(x)$ for $x \notin \mathbb{Z}_-$. Plugging back into the above:

$$\begin{aligned} \mathbb{V}\mathrm{ar}\big(E^\omega Y(2n)\big) &= \big(\tfrac{1}{8}\alpha^{-1} - \tfrac{1}{2}\big) 4\sqrt{n} \left[\frac{1}{4\sqrt{n}} + \frac{1}{2\sqrt{n}} \sum_{j=1}^{n-1} \frac{j^{-1/2}}{-\Gamma_j(-1/2)}\right] \\ &\sim \sqrt{2n} \cdot \sqrt{2}\big(\tfrac{1}{4}\alpha^{-1} - 1\big) \cdot \frac{1}{-\Gamma(-1/2)} \\ &= \sqrt{2n} \cdot \sqrt{2}\big(\tfrac{1}{4}\alpha^{-1} - 1\big) \cdot \frac{1}{2\sqrt{\pi}}. \end{aligned}$$

The theorem is proved. □

We close this section with a remark regarding the expected limit of the quenched mean process in the stationary case.

In the setting of Theorem 1.7 the random average process H from the proof of Theorem 1.6 has the initial profile

$$H_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sum_{y=1}^{x} f(T_{y,0}\omega) & \text{if } x > 0, \quad \text{and} \\ \sum_{y=x+1}^{0} E^{\omega} f(T_{y,0}\omega) & \text{if } x < 0. \end{cases}$$

Thus, to extend the convergence result of Theorem 1.6 to include the stationary case we need to prove a functional central limit theorem for the partial sums $\sum_{y=1}^{x} f(T_{y,0}\omega)$.

Finally, note that if indeed the claim of Theorem 1.6 holds in the stationary setting of Theorem 1.7, then we would have

$$\sigma_0^2 = \mathbb{V}\mathrm{ar}(f) = \frac{1}{\beta} - 1 = \frac{1}{4\alpha} - 1 = \frac{1/4 - \alpha}{\alpha} = \frac{\rho_0^2 \sigma_D^2}{\alpha}.$$

(Recall that we assumed the mean $\rho_0 = 1$ in Theorem 1.7). Thus one would have

$$\begin{aligned} \mathbf{E}[z(s,0)z(t,0)] &= \frac{\rho_0^2\sigma_D^2}{\alpha} \frac{\sigma}{\sqrt{2\pi}} (\sqrt{t} + \sqrt{s} - \sqrt{|t-s|}) \\ &= \frac{1}{2\sqrt{2\pi}} \big(\tfrac{1}{4}\alpha^{-1} - 1\big)(\sqrt{t} + \sqrt{s} - \sqrt{|t-s|}), \end{aligned}$$

as stated in Theorem 1.7.

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