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The Random Average Process and Random Walk in a Space-Time Random Environment in One Dimension

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Abstract: We study space-time fluctuations around a characteristic line for a one-dimensional interacting system known as the random average process. The state of this system is a real-valued function on the integers. New values of the function are created by averaging previous values with random weights. The fluctuations analyzed occur on the scale $n^{1/4}$, where n is the ratio of macroscopic and microscopic scales in the system. The limits of the fluctuations are described by a family of Gaussian processes. In cases of known product-form invariant distributions, this limit is a two-parameter process whose time marginals are fractional Brownian motions with Hurst parameter $1/4$. Along the way we study the limits of quenched mean processes for a random walk in a space-time random environment. These limits also happen at scale $n^{1/4}$ and are described by certain Gaussian processes that we identify. In particular, when we look at a backward quenched mean process, the limit process is the solution of a stochastic heat equation.

1. Introduction

Fluctuations for asymmetric interacting systems. An asymmetric interacting system is a random process $\sigma_\tau = \{\sigma_\tau(k) : k \in \mathcal{K}\}$ of many components $\sigma_\tau(k)$ that influence each others' evolution. Asymmetry means here that the components have an average drift in some spatial direction. Such processes are called interacting particle systems because often these components can be thought of as particles.

To orient the reader, let us first think of a single random walk $\{X_\tau : \tau = 0, 1, 2, \dots\}$ that evolves by itself. For random walk we scale both space and time by n because on this scale we see the long-term velocity: $n^{-1}X_{\lfloor nt \rfloor} \to tv$ as $n \to \infty$, where $v = EX_1$. The random walk is *diffusive* which means that its fluctuations occur on the scale $n^{1/2}$, as

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revealed by the classical central limit theorem: $n^{-1/2}(X_{\lfloor nt \rfloor} - ntv)$ converges weakly to a Gaussian distribution. The Gaussian limit is *universal* here because it arises regardless of the choice of step distribution for the random walk, as long as a square-integrability hypothesis is satisfied.

For asymmetric interacting systems we typically also scale time and space by the same factor n, and this is known as *Euler scaling.* However, in certain classes of one-dimensional asymmetric interacting systems the random evolution produces fluctuations of smaller order than the natural diffusive scale. Two types of such phenomena have been discovered.

(i) In Hammersley's process, in asymmetric exclusion, and in some other closely related systems, dynamical fluctuations occur on the scale $n^{1/3}$. Currently known rigorous results suggest that the Tracy-Widom distributions from random matrix theory are the universal limits of these $n^{1/3}$ fluctuations.

The seminal works in this context are by Baik, Deift and Johansson [3] on Hammersley's process and by Johansson [19] on the exclusion process. We should point out though that [3] does not explicitly discuss Hammersley's process, but instead the maximal number of planar Poisson points on an increasing path in a rectangle. One can intrepret the results in [3] as fluctuation results for Hammersley's process with a special initial configuration. The connection between the increasing path model and Hammersley's process goes back to Hammersley's paper [18]. It was first utilized by Aldous and Diaconis [1] (who also named the process), and then further in the papers [26, 28].

(ii) The second type has fluctuations of the order $n^{1/4}$ and limits described by a family of self-similar Gaussian processes that includes fractional Brownian motion with Hurst parameter $\frac{1}{4}$. This result was first proved for a system of independent random walks [30]. One of the main results of the current paper shows that the $n^{1/4}$ fluctuations also appear in a family of interacting systems called *random average processes* in one dimension. The same family of limiting Gaussian processes appears here too, suggesting that these limits are universal for some class of interacting systems.

The random average processes (RAP) studied in the present paper describe a random real-valued function on the integers whose values evolve by jumping to random convex combinations of values in a finite neighborhood. It could be thought of as a caricature model for an interface between two phases on the plane, hence we call the state a *height function*. RAP is related to the so-called linear systems discussed in Chapter IX of Liggett's monograph [22]. RAP was introduced by Ferrari and Fontes [14] who studied the fluctuations from initial linear slopes. In particular, they discovered that the height over the origin satisfies a central limit theorem in the time scale $t^{1/4}$. The Ferrari-Fontes results suggested RAP to us as a fruitful place to investigate whether the $n^{1/4}$ fluctuation picture discovered in [30] for independent walks had any claim to universality.

There are two ways to see the lower order dynamical fluctuations.

(1) One can take deterministic initial conditions so that only dynamical randomness is present.
(2) Even if the initial state is random with central limit scale fluctuations, one can find the lower order fluctuations by looking at the evolution of the process along a characteristic curve.

Articles [3] and [19] studied the evolutions of special deterministic initial states of Hammersley's process and the exclusion process. Recently Ferrari and Spohn [15] have extended this analysis to the fluctuations across a characteristic in a stationary exclusion process. The general nonequilibrium hydrodynamic limit situation is still out of reach for these models. [30] contains a tail bound for Hammersley's process that suggests

$n^{1/3}$ scaling also in the nonequilibrium situation, including along a shock which can be regarded as a "generalized" characteristic.

Our results for the random average process are for the general hydrodynamic limit setting. The initial increments of the random height function are assumed independent and subject to some moment bounds. Their means and variances must vary sufficiently regularly to satisfy a Hölder condition. Deterministic initial increments qualify here as a special case of independent.

The classification of the systems mentioned above (Hammersley, exclusion, independent walks, RAP) into $n^{1/3}$ and $n^{1/4}$ fluctuations coincides with their classification according to type of macroscopic equation. Independent particles and RAP are macroscopically governed by linear first-order partial differential equations $u_t + bu_x = 0$. In contrast, macroscopic evolutions of Hammersley's process and the exclusion process obey genuinely nonlinear Hamilton-Jacobi equations $u_t + f(u_x) = 0$ that create shocks.

Suppose we start off one of these systems so that the initial state fluctuates on the $n^{1/2}$ spatial scale, for example in a stationary distribution. Then the fluctuations of the entire system on the $n^{1/2}$ scale simply consist of initial fluctuations transported along the deterministic characteristics of the macroscopic equation. This is a consequence of the lower order of dynamical fluctuations. When the macroscopic equation is linear this is the whole picture of diffusive fluctuations. In the nonlinear case the behavior at the shocks (where characteristics merge) also needs to be resolved. This has been done for the exclusion process [25] and for Hammersley's process [29].

Random walk in a space-time random environment. Analysis of the random average process utilizes a dual description in terms of backward random walks in a space-time random environment. Investigation of the fluctuations of RAP leads to a study of fluctuations of these random walks, both quenched invariance principles for the walk itself and limits for the quenched mean process. The quenched invariance principles have been reported elsewhere [24]. The results for the quenched mean process are included in the present paper because they are intimately connected to the random average process results.

We look at two types of processes of quenched means. We call them forward and backward. In the forward case the initial point of the walk is fixed, and the walk runs for a specified amount of time on the space-time lattice. In the backward case the initial point moves along a characteristic, and the walk runs until it reaches the horizontal axis. Furthermore, in both cases we let the starting point vary horizontally (spatially), and so we have a space-time process. In both cases we describe a limiting Gaussian process, when space is scaled by $n^{1/2}$, time by n, and the magnitude of the fluctuations by $n^{1/4}$. In particular, in the backward case we find a limit process that solves the stochastic heat equation.

There are two earlier papers on the quenched mean of this random walk in a space-time random environment. These previous results were proved under assumptions of small enough noise and finitely many possible values for the random probabilities. Bernabei [5] showed that the centered quenched mean, normalized by its own standard deviation, converges to a normal variable. Then separately he showed that this standard deviation is bounded above and below on the order $n^{1/4}$. Bernabei has results also in dimension 2, and also for the quenched covariance of the walk. Boldrighini and Pellegrinotti [6] also proved a normal limit in the scale $n^{1/4}$ for what they term the "correction" caused by the random environment on the mean of a test function.

Finite-dimensional versus process-level convergence. Our main results all state that the finite-dimensional distributions of a process of interest converge to the finite-dimensional distributions of a certain Gaussian process specified by its covariance function. We have not proved process-level tightness, except in the case of forward quenched means for the random walks where we compute a bound on the sixth moment of the process increment.

Further relevant literature. It is not clear what exactly are the systems "closely related" to Hammersley's process or exclusion process, alluded to in the beginning of the Introduction, that share the $n^{1/3}$ fluctuations and Tracy-Widom limits. The processes for which rigorous proofs exist all have an underlying representation in terms of a last-passage percolation model. Another such example is "oriented digital boiling" studied by Gravner, Tracy and Widom [16]. (This model was studied earlier in [27] and [20] under different names.)

Fluctuations of the current were initially studied from the perspective of a moving observer traveling with a general speed. The fluctuations are diffusive, and the limiting variance is a function of the speed of the observer. The special nature of the characteristic speed manifests itself in the vanishing of the limiting variance on this diffusive scale. The early paper of Ferrari and Fontes [13] treated the asymmetric exclusion process. Their work was extended by Balázs [4] to a class of deposition models that includes the much-studied zero range process and a generalization called the bricklayers' process.

Work on the fluctuations of Hammersley's process and the exclusion process has connections to several parts of mathematics. Overviews of some of these links appear in papers [2, 10, 17]. General treatments of large scale behavior of interacting random systems can be found in [9, 21–23, 32, 33].

Organization of the paper. We begin with the description of the random average process and the limit theorem for it in Sect. 2. Section 3 describes the random walk in a space-time random environment and the limit theorems for quenched mean processes. The proofs begin with Sect. 4 that lays out some preliminary facts on random walks. Sections 5 and 6 prove the fluctuation results for random walk, and the final Sect. 7 proves the limit theorem for RAP.

The reader only interested in the random walk can read Sect. 3 and the proofs for the random walk limits independently of the rest of the paper, except for certain definitions and a hypothesis which have been labeled. The RAP results can be read independently of the random walk, but their proofs depend on the random walk results.

Notation. We summarize here some notation and conventions for quick reference. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$, while $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$ and $\mathbb{R}_+ = [0, \infty)$. On the two dimensional integer lattice $\mathbb{Z}^2$ standard basis vectors are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The e_2-direction represents time.

We need several different probability measures and corresponding expectation operators. $\mathbb{P}$ (with expectation $\mathbb{E}$) is the probability measure on the space Ω of environments ω. $\mathbb{P}$ is an i.i.d. product measure across the coordinates indexed by the space-time lattice $\mathbb{Z}^2$. $\mathbf{P}$ (with expectation $\mathbf{E}$) is the probability measure of the initial state of the random average process. $\mathbf{E}^\omega$ is used to emphasize that an expectation over initial states is taken with a fixed environment ω. Jointly the environment and initial state are independent, so the joint measure is the product $\mathbb{P} \otimes \mathbf{P}$. P^ω (with expectation E^ω) is the quenched path measure of the random walks in environment ω. The annealed measure for the walks is $P = \int P^\omega \, \mathbb{P}(d\omega)$. Additionally, we use P and E for generic probability measures and expectations for processes that are not part of this specific set-up, such as Brownian motions and limiting Gaussian processes.

The environments $\omega \in \Omega$ are configurations $\omega = (\omega_{x,\tau} : (x, \tau) \in \mathbb{Z}^2)$ of vectors indexed by the space-time lattice $\mathbb{Z}^2$. Each element $\omega_{x,\tau}$ is a probability vector of length $2M + 1$, denoted also by $u_\tau(x) = \omega_{x,\tau}$, and in terms of coordinates $u_\tau(x) = (u_\tau(x, y) : -M \leq y \leq M)$. The environment at a fixed time value τ is $\bar{\omega}_\tau = (\omega_{x,\tau} : x \in \mathbb{Z})$. Translations on Ω are defined by $(T_{x,\tau}\omega)_{y,s} = \omega_{x+y,\tau+s}$.

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ is the lower integer part of a real x. Throughout, C denotes a constant whose exact value is immaterial and can change from line to line. The density and cumulative distribution function of the centered Gaussian distribution with variance σ^2 are denoted by $\varphi_{\sigma^2}(x)$ and $\Phi_{\sigma^2}(x)$. $\{B(t) : t \geq 0\}$ is one-dimensional standard Brownian motion, in other words the Gaussian process with covariance $EB(s)B(t) = s \wedge t$.

2. The Random Average Process

The state of the random average process (RAP) is a height function $\sigma : \mathbb{Z} \to \mathbb{R}$. It can also be thought of as a sequence $\sigma = (\sigma(i) : i \in \mathbb{Z}) \in \mathbb{R}^{\mathbb{Z}}$, where $\sigma(i)$ is the height of an interface above site i. The state evolves in discrete time according to the following rule. At each time point $\tau = 1, 2, 3, \ldots$ and at each site $k \in \mathbb{Z}$, a random probability vector $u_\tau(k) = (u_\tau(k, j) : -M \leq j \leq M)$ of length $2M + 1$ is drawn. Given the state $\sigma_{\tau-1} = (\sigma_{\tau-1}(i) : i \in \mathbb{Z})$ at time $\tau - 1$, the height value at site k is then updated to

$$\sigma_\tau(k) = \sum_{j:|j|\leq M} u_\tau(k, j)\sigma_{\tau-1}(k + j). \tag{2.1}$$

This update is performed independently at each site k to form the state $\sigma_\tau = (\sigma_\tau(k) : k \in \mathbb{Z})$ at time τ. The same step is repeated at the next time $\tau + 1$ with new independent draws of the probability vectors.

So, given an initial state σ_0, the process σ_τ is constructed with a collection $\{u_\tau(k) : \tau \in \mathbb{N}, k \in \mathbb{Z}\}$ of independent and identically distributed random vectors. These random vectors are defined on a probability space $(\Omega, \mathfrak{S}, \mathbb{P})$. If σ_0 is also random with distribution $\mathbf{P}$, then σ_0 and the vectors $\{u_\tau(k)\}$ are independent, in other words the joint distribution is $\mathbb{P} \otimes \mathbf{P}$. We write $u_\tau^\omega(k)$ to make explicit the dependence on $\omega \in \Omega$. $\mathbb{E}$ will denote expectation under the measure $\mathbb{P}$. M is the *range* and is a fixed finite parameter of the model. $\mathbb{P}$-almost surely each random vector $u_\tau(k)$ satisfies

$$0 \leq u_\tau(k, j) \leq 1 \text{ for all } -M \leq j \leq M, \text{ and } \sum_{j=-M}^{M} u_\tau(k, j) = 1.$$

It is often convenient to allow values $u_\tau(k, j)$ for all j. Then automatically $u_\tau(k, j) = 0$ for $|j| > M$.

Let

$$p(0, j) = \mathbb{E}u_0(0, j)$$

denote the averaged probabilities. Throughout the paper we make two fundamental assumptions.

(i) First, there is no integer $h > 1$ such that, for some $x \in \mathbb{Z}$,

$$\sum_{k \in \mathbb{Z}} p(0, x + kh) = 1.$$

This is also expressed by saying that the *span* of the random walk with jump probabilities $p(0, j)$ is 1 [11, p. 129]. It follows that the group generated by $\{x \in \mathbb{Z} : p(0, x) > 0\}$ is all of $\mathbb{Z}$, in other words this walk is *aperiodic* in Spitzer's terminology [31].

(ii) Second, we assume that

$$\mathbb{P}\{\max_j u_0(0, j) < 1\} > 0. \tag{2.2}$$

If this assumption fails, then $\mathbb{P}$-almost surely for each (k, τ) there exists $j = j(k, \tau)$ such that $u_\tau(k, j) = 1$. No averaging happens, but instead $\sigma_\tau(k)$ adopts the value $\sigma_{\tau-1}(k+j)$. The behavior is then different from that described by our results.

No further hypotheses are required of the distribution $\mathbb{P}$ on the probability vectors. Deterministic weights $u^\omega_\tau(k, j) \equiv p(0, j)$ are also admissible, in which case (2.2) requires $\max_j p(0, j) < 1$.

In addition to the height process σ_τ we also consider the increment process $\eta_\tau = (\eta_\tau(i) : i \in \mathbb{Z})$ defined by

$$\eta_\tau(i) = \sigma_\tau(i) - \sigma_\tau(i - 1).$$

From (2.1) one can deduce a similar linear equation for the evolution of the increment process. However, the weights are not necessarily nonnegative, and even if they are, they do not necessarily sum to one.

Next we define several constants that appear in the results.

$$D(\omega) = \sum_{x \in \mathbb{Z}} x\, u^\omega_0(0, x) \tag{2.3}$$

is the drift at the origin. Its mean is $V = \mathbb{E}(D)$ and variance

$$\sigma_D^2 = \mathbb{E}[(D - V)^2]. \tag{2.4}$$

A variance under averaged probabilities is computed by

$$\sigma_a^2 = \sum_{x \in \mathbb{Z}} (x - V)^2\, p(0, x). \tag{2.5}$$

Define random and averaged characteristic functions by

$$\phi^\omega(t) = \sum_{x \in \mathbb{Z}} u^\omega_0(0, x) e^{itx} \quad \text{and} \quad \phi_a(t) = \mathbb{E}\phi^\omega(t) = \sum_{x \in \mathbb{Z}} p(0, x) e^{itx}, \tag{2.6}$$

and then further

$$\lambda(t) = \mathbb{E}[\, |\phi^\omega(t)|^2\,] \quad \text{and} \quad \bar{\lambda}(t) = |\phi_a(t)|^2. \tag{2.7}$$

Finally, define a positive constant β by

$$\beta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \lambda(t)}{1 - \bar{\lambda}(t)}\, dt. \tag{2.8}$$

The assumption of span 1 implies that $|\phi_a(t)| = 1$ only at multiples of 2π. Hence the integrand above is positive at $t \neq 0$. Separately one can check that the integrand has a finite limit as $t \to 0$. Thus β is well-defined and finite.

In Sect. 4 we can give these constants, especially β, more probabilistic meaning from the perspective of the underlying random walk in random environment.

For the limit theorems we consider a sequence σ_τ^n of the random average processes, indexed by $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Initially we set $\sigma_0^n(0) = 0$. For each n we assume that the initial increments $\left\{\eta_0^n(i) : i \in \mathbb{Z}\right\}$ are independent random variables, with

$$\mathbf{E}[\eta_0^n(i)] = \varrho(i/n) \quad \text{and} \quad \mathbf{Var}[\eta_0^n(i)] = v(i/n). \tag{2.9}$$

The functions ϱ and v that appear above are assumed to be uniformly bounded functions on $\mathbb{R}$ and to satisfy this local Hölder continuity:

$$\begin{aligned}&\text{For each compact interval } [a,b] \subseteq \mathbb{R} \text{ there exist}\\ &C = C(a,b) < \infty \text{ and } \gamma = \gamma(a,b) > 1/2 \text{ such that}\\ &|\varrho(x) - \varrho(y)| + |v(x) - v(y)| \le C\,|x-y|^\gamma \quad \text{for } x, y \in [a,b].\end{aligned} \tag{2.10}$$

The function v must be nonnegative, but the sign of ϱ is not restricted. Both functions are allowed to vanish. In particular, our hypotheses permit deterministic initial heights which implies that v vanishes identically.

The distribution on initial heights and increments described above is denoted by $\mathbf{P}$. We make this uniform moment hypothesis on the increments:

$$\text{there exists } \alpha > 0 \text{ such that} \quad \sup_{n \in \mathbb{N},\, i \in \mathbb{Z}} \mathbf{E}\left[\,|\eta_0^n(i)|^{2+\alpha}\,\right] < \infty. \tag{2.11}$$

We assume that the processes σ_τ^n are all defined on the same probability space. The environments ω that drive the dynamics are independent of the initial states $\{\sigma_0^n\}$, so the joint distribution of $(\omega, \{\sigma_0^n\})$ is $\mathbb{P} \otimes \mathbf{P}$. When computing an expectation under a fixed ω we write $\mathbf{E}^\omega$.

On the larger space and time scale the height function is simply rigidly translated at speed $b = -V$, and the same is also true of the central limit fluctuations of the initial height function. Precisely speaking, define a function U on $\mathbb{R}$ by $U(0) = 0$ and $U'(x) = \varrho(x)$. Let $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. The assumptions made thus far imply that both

$$n^{-1}\sigma_{\lfloor nt \rfloor}^n(\lfloor nx \rfloor) \longrightarrow U(x - bt) \tag{2.12}$$

and

$$\frac{\sigma_{\lfloor nt \rfloor}^n(\lfloor nx \rfloor) - nU(x-bt)}{\sqrt{n}} - \frac{\sigma_0^n(\lfloor nx \rfloor - \lfloor nbt \rfloor) - nU(x-bt)}{\sqrt{n}} \longrightarrow 0 \tag{2.13}$$

in probability, as $n \to \infty$. (We will not give a proof. This follows from easier versions of the estimates in the paper.) Limit (2.12) is the "hydrodynamic limit" of the process. The large scale evolution of the height process is thus governed by the linear transport equation

$$w_t + bw_x = 0.$$

This equation is uniquely solved by $w(x,t) = U(x - bt)$ given the initial function $w(x,0) = U(x)$. The lines $x(t) = x + bt$ are the characteristics of this equation, the curves along which the equation carries information. Limit (2.13) says that fluctuations on the diffusive scale do not include any randomness from the evolution, only a translation of initial fluctuations along characteristics.

We find interesting height fluctuations along a macroscopic characteristic line $x(t) = \bar{y} + bt$, and around such a line on the microscopic spatial scale $\sqrt{n}$. The magnitude of these fluctuations is of the order $n^{1/4}$, so we study the process

$$z_n(t,r) = n^{-1/4}\big\{\sigma^n_{\lfloor nt\rfloor}(\lfloor n\bar{y}\rfloor + \lfloor r\sqrt{n}\,\rfloor + \lfloor ntb\rfloor) - \sigma^n_0(\lfloor n\bar{y}\rfloor + \lfloor r\sqrt{n}\,\rfloor)\big\},$$

indexed by $(t,r) \in \mathbb{R}_+ \times \mathbb{R}$, for a fixed $\bar{y} \in \mathbb{R}$. In terms of the increment process η^n_τ, $z_n(t,0)$ is the net flow from right to left across the discrete characteristic $\lfloor n\bar{y}\rfloor + \lfloor nsb\rfloor$, during the time interval $0 \le s \le t$.

Next we describe the limit of z_n. Recall the constants defined in (2.4), (2.5), and (2.8). Combine them into a new constant

$$\kappa = \frac{\sigma_D^2}{\beta\sigma_a^2}. \tag{2.14}$$

Let $\{B(t) : t \ge 0\}$ be one-dimensional standard Brownian motion. Define two functions Γ_q and Γ_0 on $(\mathbb{R}_+ \times \mathbb{R}) \times (\mathbb{R}_+ \times \mathbb{R})$:

$$\Gamma_q((s,q),(t,r)) = \frac{\kappa}{2}\int_{\sigma_a^2|t-s|}^{\sigma_a^2(t+s)} \frac{1}{\sqrt{2\pi v}}\exp\Big\{-\frac{1}{2v}(q-r)^2\Big\}\,dv \tag{2.15}$$

and

$$\begin{aligned}\Gamma_0((s,q),(t,r)) = &\int_{q\vee r}^{\infty} P[\sigma_a B(s) > x-q]P[\sigma_a B(t) > x-r]\,dx\\ &-\Big\{\mathbf{1}_{\{r>q\}}\int_q^r P[\sigma_a B(s) > x-q]P[\sigma_a B(t) \le x-r]\,dx\\ &+\mathbf{1}_{\{q>r\}}\int_r^q P[\sigma_a B(s) \le x-q]P[\sigma_a B(t) > x-r]\,dx\Big\}\\ &+\int_{-\infty}^{q\wedge r} P[\sigma_a B(s) \le x-q]P[\sigma_a B(t) \le x-r]\,dx. \end{aligned} \tag{2.16}$$

The boundary values are such that $\Gamma_q((s,q),(t,r)) = \Gamma_0((s,q),(t,r)) = 0$ if either $s = 0$ or $t = 0$. We will see later that Γ_q is the limiting covariance of the backward quenched mean process of a related random walk in random environment. Γ_0 is the covariance for fluctuations contributed by the initial increments of the random average process. (Hence the subscripts q for quenched and 0 for initial time. The subscript on Γ_q has nothing to do with the argument (s,q).)

The integral expressions above are the form in which Γ_q and Γ_0 appear in the proofs. For Γ_q the key point is the limit (5.19) which is evaluated earlier in (4.5). Γ_0 arises in Proposition 7.1.

Here are alternative succinct representations for Γ_q and Γ_0. Denote the centered Gaussian density with variance σ^2 by

$$\varphi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big\{-\frac{1}{2\sigma^2}x^2\Big\} \tag{2.17}$$

and its distribution function by $\Phi_{\sigma^2}(x) = \int_{-\infty}^{x} \varphi_{\sigma^2}(y)\,dy$. Then define

$$\Psi_{\sigma^2}(x) = \sigma^2 \varphi_{\sigma^2}(x) - x(1 - \Phi_{\sigma^2}(x)),$$

which is an antiderivative of $\Phi_{\sigma^2}(x) - 1$. In these terms,

$$\Gamma_q((s,q),(t,r)) = \kappa \Psi_{\sigma_a^2(t+s)}\big(|q-r|\big) - \kappa \Psi_{\sigma_a^2|t-s|}\big(|q-r|\big)$$

and

$$\Gamma_0((s,q),(t,r)) = \Psi_{\sigma_a^2 s}\big(|q-r|\big) + \Psi_{\sigma_a^2 t}\big(|q-r|\big) - \Psi_{\sigma_a^2(t+s)}\big(|q-r|\big).$$

Theorem 2.1. *Assume* (2.2) *and that the averaged probabilities* $p(0,j) = \mathbb{E}u_0^\omega(0,j)$ *have lattice span 1. Let* ϱ *and* v *be two uniformly bounded functions on* $\mathbb{R}$ *that satisfy the local Hölder condition* (2.10). *For each n, let* σ_τ^n *be a random average process normalized by* $\sigma_0^n(0) = 0$ *and whose initial increments* $\{\eta_0^n(i) : i \in \mathbb{Z}\}$ *are independent and satisfy* (2.9) *and* (2.11). *Assume the environments* ω *independent of the initial heights* $\{\sigma_0^n : n \in \mathbb{N}\}$.

Fix $\bar{y} \in \mathbb{R}$. *Under the above assumptions the finite-dimensional distributions of the process* $\{z_n(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$ *converge weakly as* $n \to \infty$ *to the finite-dimensional distributions of the mean zero Gaussian process* $\{z(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$ *specified by the covariance*

$$Ez(s,q)z(t,r) = \varrho(\bar{y})^2 \Gamma_q((s,q),(t,r)) + v(\bar{y})\Gamma_0((s,q),(t,r)). \tag{2.18}$$

The statement means that, given space-time points $(t_1,r_1),\ldots,(t_k,r_k)$, the $\mathbb{R}^k$-valued random vector $(z_n(t_1,r_1),\ldots,z_n(t_k,r_k))$ converges in distribution to the random vector $(z(t_1,r_1),\ldots,z(t_k,r_k))$ as $n \to \infty$. The theorem is also valid in cases where one source of randomness has been turned off: if initial increments around $\lfloor n\bar{y} \rfloor$ are deterministic then $v(\bar{y}) = 0$, while if $D(\omega) \equiv V$ then $\sigma_D^2 = 0$. The case $\sigma_D^2 = 0$ contains as a special case the one with deterministic weights $u_\tau^\omega(k,j) \equiv p(0,j)$.

If we consider only temporal correlations with a fixed r, the formula for the covariance is as follows:

$$\begin{aligned} Ez(s,r)z(t,r) = {} & \frac{\kappa\sigma_a}{\sqrt{2\pi}} \varrho(\bar{y})^2 \big(\sqrt{s+t} - \sqrt{t-s}\,\big) \\ & + \frac{\sigma_a}{\sqrt{2\pi}} v(\bar{y}) \big(\sqrt{s} + \sqrt{t} - \sqrt{s+t}\,\big) \quad \text{for } s < t. \end{aligned} \tag{2.19}$$

Remark 2.1. The covariances are central to our proofs but they do not illuminate the behavior of the process z. Here is a stochastic integral representation of the Gaussian process with covariance (2.18):

$$\begin{aligned} z(t,r) = {} & \varrho(\bar{y})\sigma_a\sqrt{\kappa} \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma_a^2(t-s)}(r-x)\,dW(s,x) \\ & + \sqrt{v(\bar{y})} \int_{\mathbb{R}} \operatorname{sign}(x-r)\Phi_{\sigma_a^2 t}\big(-|x-r|\big)\,dB(x). \end{aligned} \tag{2.20}$$

Above W is a two-parameter Brownian motion defined on $\mathbb{R}_+ \times \mathbb{R}$, B is a one-parameter Brownian motion defined on $\mathbb{R}$, and W and B are independent of each other. The first integral represents the space-time noise created by the dynamics, and the second integral represents the initial noise propagated by the evolution. The equality in (2.20) is equality in distribution of processes. It can be verified by checking that the Gaussian process defined by the sum of the integrals has the covariance (2.18).

One can readily see the second integral in (2.20) arise as a sum in the proof. It is the limit of $Y^n(t,r)$ defined below Eq. (7.1).

One can also check that the right-hand side of (2.20) is a weak solution of a stochastic heat equation with two independent sources of noise:

$$z_t = \tfrac{1}{2}\sigma_a^2 z_{rr} + \varrho(\bar{y})\sigma_a\sqrt{\kappa}\,\dot{W} + \tfrac{1}{2}\sqrt{v(\bar{y})}\sigma_a^2 B'', \qquad z(0,r) \equiv 0. \tag{2.21}$$

$\dot{W}$ is space-time white noise generated by the dynamics and B'' the second derivative of the one-dimensional Brownian motion that represents initial noise. This equation has to be interpreted in a weak sense through integration against smooth compactly supported test functions. We make a related remark below in Sect. 3.2 for limit processes of quenched means of space-time RWRE.

The simplest RAP dynamics averages only two neighboring height values. By translating the indices, we can assume that $p(0,-1) + p(0,0) = 1$. In this case the evolution of increments is given by the equation

$$\eta_\tau(k) = u_\tau(k,0)\eta_{\tau-1}(k) + u_\tau(k-1,-1)\eta_{\tau-1}(k-1). \tag{2.22}$$

There is a queueing interpretation of sorts for this evolution. Suppose $\eta_{\tau-1}(k)$ denotes the amount of work that remains at station k at the end of cycle $\tau-1$. Then during cycle τ, the fraction $u_\tau(k,-1)$ of this work is completed and moves on to station $k+1$, while the remaining fraction $u_\tau(k,0)$ stays at station k for further processing.

In this case we can explicitly evaluate the constant β in terms of the other quantities. In a particular stationary situation we can also identify the temporal marginal of z in (2.19) as a familiar process. (A probability distribution μ on the space $\mathbb{Z}^{\mathbb{Z}}$ is an invariant distribution for the increment process if it is the case that when η_0 has μ distribution, so does η_τ for all times $\tau \in \mathbb{Z}_+$.)

Proposition 2.2. *Assume $p(0,-1) + p(0,0) = 1$.*

(a) *Then*

$$\beta = \frac{1}{\sigma_a^2}\mathbb{E}[u_0(0,0)u_0(0,-1)]. \tag{2.23}$$

(b) *Suppose further that the increment process η_τ possesses an invariant distribution μ in which the variables $\{\eta(i) : i \in \mathbb{Z}\}$ are i.i.d. with common mean $\varrho = E^\mu[\eta(i)]$ and variance $v = E^\mu[\eta(i)^2] - \varrho^2$. Then $v = \kappa\varrho^2$.*

Suppose that in Theorem 2.1 each $\eta_\tau^n = \eta_\tau$ is a stationary process with marginal μ. Then the limit process z has covariance

$$Ez(s,q)z(t,r) = \kappa\varrho^2\Big(\Psi_{\sigma_a^2 s}\big(|q-r|\big) + \Psi_{\sigma_a^2 t}\big(|q-r|\big) - \Psi_{\sigma_a^2|t-s|}\big(|q-r|\big)\Big). \tag{2.24}$$

In particular, for a fixed r the process $\{z(t,r) : t \in \mathbb{R}_+\}$ has covariance

$$Ez(s,r)z(t,r) = \frac{\sigma_a\kappa\varrho^2}{\sqrt{2\pi}}\big(\sqrt{s} + \sqrt{t} - \sqrt{|t-s|}\big). \tag{2.25}$$

In other words, process $z(\cdot,r)$ is fractional Brownian motion with Hurst parameter $1/4$.

To rephrase the connection (2.24)–(2.25), the process $\{z(t,r)\}$ in (2.24) is a certain two-parameter process whose marginals along the first parameter direction are fractional Brownian motions.

Ferrari and Fontes [14] showed that given any slope ρ, the process η_τ started from deterministic increments $\eta_0(x) = \rho x$ converges weakly to an invariant distribution. But as is typical for interacting systems, there is little information about the invariant distributions in the general case. The next example gives a family of processes and i.i.d. invariant distributions to show that part (b) of Proposition 2.2 is not vacuous. Presently we are not aware of other explictly known invariant distributions for RAP.

Example 2.1. Fix integer parameters $m > j > 0$. Let $\{u_\tau(k,-1) : \tau \in \mathbb{N}, k \in \mathbb{Z}\}$ be i.i.d. beta-distributed random variables with density

$$h(u) = \frac{(m-1)!}{(j-1)!(m-j-1)!} u^{j-1}(1-u)^{m-j-1}$$

on $(0,1)$. Set $u_\tau(k,0) = 1 - u_\tau(k,-1)$. Consider the evolution defined by (2.22) with these weights. Then a family of invariant distributions for the increment process $\eta_\tau = (\eta_\tau(k) : k \in \mathbb{Z})$ is obtained by letting the variables $\{\eta(k)\}$ be i.i.d. gamma distributed with common density

$$f(x) = \frac{1}{(m-1)!} \lambda e^{-\lambda x} (\lambda x)^{m-1} \tag{2.26}$$

on $\mathbb{R}_+$. The family of invariant distributions is parametrized by $0 < \lambda < \infty$. Under this distribution $\mathbf{E}[\eta(k)] = m/\lambda$ and $\mathbf{Var}[\eta(k)] = m/\lambda^2$.

One motivation for the present work was to investigate whether the limits found in [30] for fluctuations along a characteristic for independent walks are instances of some universal behavior. The present results are in agreement with those obtained for independent walks. The common scaling is $n^{1/4}$. In that paper only the case $r = 0$ of Theorem 2.1 was studied. For both independent walks and RAP the limit $z(\cdot\,,0)$ is a mean-zero Gaussian process with covariance of the type

$$Ez(s,0)z(t,0) = c_1\left(\sqrt{s+t} - \sqrt{t-s}\,\right) + c_2\left(\sqrt{s} + \sqrt{t} - \sqrt{s+t}\,\right),$$

where c_1 is determined by the mean increment and c_2 by the variance of the increment locally around the initial point of the characteristic. Furthermore, as in Proposition 2.2(b), for independent walks the limit process specializes to fractional Brownian motion if the increment process is stationary.

These and other related results suggest several avenues of inquiry. In the introduction we contrasted this picture of $n^{1/4}$ fluctuations and fractional Brownian motion limits with the $n^{1/3}$ fluctuations and Tracy-Widom limits found in exclusion and Hammersley processes. Obviously more classes of processes should be investigated to understand better the demarcation between these two types. Also, there might be further classes with different limits.

Above we assumed independent increments at time zero. It would be of interest to see if relaxing this assumption leads to a change in the second part of the covariance (2.18). [The first part comes from the random walks in the dual description and would not be affected by the initial conditions.] However, without knowledge of some explicit invariant distributions it is not clear what types of initial increment processes $\{\eta_0(k)\}$ are worth considering. Unfortunately finding explicit invariant distributions for interacting systems seems often a matter of good fortune.

We conclude this section with the dual description of RAP which leads us to study random walks in a space-time random environment. Given ω, let $\{X_s^{i,\tau} : s \in \mathbb{Z}_+\}$ denote a random walk on $\mathbb{Z}$ that starts at $X_0^{i,\tau} = i$, and whose transition probabilities are given by

$$P^\omega(X_{s+1}^{i,\tau} = y \mid X_s^{i,\tau} = x) = u_{\tau-s}^\omega(x, y-x). \tag{2.27}$$

P^ω is the path measure of the walk $X_s^{i,\tau}$, with expectation denoted by E^ω. Comparison of (2.1) and (2.27) gives

$$\sigma_\tau(i) = \sum_j P^\omega(X_1^{i,\tau} = j \mid X_0^{i,\tau} = i)\sigma_{\tau-1}(j) = E^\omega\big[\sigma_{\tau-1}(X_1^{i,\tau})\big]. \tag{2.28}$$

Iteration and the Markov property of the walks $X_s^{i,\tau}$ then lead to

$$\sigma_\tau(i) = E^\omega\big[\sigma_0(X_\tau^{i,\tau})\big]. \tag{2.29}$$

Note that the initial height function σ_0 is a constant under the expectation E^ω.

Let us add another coordinate to keep track of time and write $\bar{X}_s^{i,\tau} = (X_s^{i,\tau}, \tau - s)$ for $s \geq 0$. Then $\bar{X}_s^{i,\tau}$ is a random walk on the planar lattice $\mathbb{Z}^2$ that always moves down one step in the e_2-direction, and if its current position is (x, n), the e_1-coordinate of its next position is $x + y$ with probability $u_n(x, y)$. We shall call it the backward random walk in a random environment. In the next section we discuss this walk and its forward counterpart.

3. Random Walk in a Space-Time Random Environment

3.1. Definition of the model. We consider here a particular random walk in random environment (RWRE). The walk evolves on the planar integer lattice $\mathbb{Z}^2$, which we think of as space-time: the first component represents one-dimensional discrete space, and the second represents discrete time. We denote by e_2 the unit vector in the time-direction. The walks will not be random in the e_2-direction, but only in the spatial e_1-direction.

We consider *forward walks* $\bar{Z}_m^{i,\tau}$ and *backward walks* $\bar{X}_m^{i,\tau}$. The subscript $m \in \mathbb{Z}_+$ is the time parameter of the walk and superscripts are initial points:

$$\bar{Z}_0^{i,\tau} = \bar{X}_0^{i,\tau} = (i, \tau) \in \mathbb{Z}^2. \tag{3.1}$$

The forward walks move deterministically up in time, while the backward walks move deterministically down in time:

$$\bar{Z}_m^{i,\tau} = (Z_m^{i,\tau}, \tau + m) \quad \text{and} \quad \bar{X}_m^{i,\tau} = (X_m^{i,\tau}, \tau - m) \quad \text{for } m \geq 0.$$

Since the time components of the walks are deterministic, only the spatial components $Z_m^{i,\tau}$ and $X_m^{i,\tau}$ are really relevant. We impose a finite range on the steps of the walks: there is a fixed constant M such that

$$\left|Z_{m+1}^{i,\tau} - Z_m^{i,\tau}\right| \leq M \quad \text{and} \quad \left|X_{m+1}^{i,\tau} - X_m^{i,\tau}\right| \leq M. \tag{3.2}$$

A note of advance justification for the setting: The backward walks are the ones relevant to the random average process. Distributions of forward and backward walks are obvious mappings of each other. However, we will be interested in the quenched mean processes of the walks as we vary the final time for the forward walk or the initial space-time point for the backward walk. The results for the forward walk form an interesting point of comparison to the backward walk, even though they will not be used to analyze the random average process.

An *environment* is a configuration of probability vectors $\omega = \big(u_\tau(x) : (x, \tau) \in \mathbb{Z}^2\big)$, where each vector $u_\tau(x) = (u_\tau(x, y) : -M \le y \le M)$ satisfies

$$0 \le u_\tau(x, y) \le 1 \text{ for all } -M \le y \le M, \text{ and } \quad \sum_{y=-M}^{M} u_\tau(x, y) = 1.$$

An environment ω is a sample point of the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$. The sample space is the product space $\Omega = \mathcal{P}^{\mathbb{Z}^2}$, where $\mathcal{P}$ is the space of probability vectors of length $2M + 1$, and $\mathfrak{S}$ is the product σ-field on Ω induced by the Borel sets on $\mathcal{P}$. Throughout, we assume that $\mathbb{P}$ is a product probability measure on Ω such that the vectors $\{u_\tau(x) : (x, \tau) \in \mathbb{Z}^2\}$ are independent and identically distributed. Expectation under $\mathbb{P}$ is denoted by $\mathbb{E}$. When for notational convenience we wish to think of $u_\tau(x)$ as an infinite vector, then $u_\tau(x, y) = 0$ for $|y| > M$. We write $u_\tau^\omega(x, y)$ to make explicit the environment ω, and also $\omega_{x,\tau} = u_\tau(x)$ for the environment at space-time point (x, τ).

Fix an environment ω and an initial point (i, τ). The forward and backward walks $\bar{Z}_m^{i,\tau}$ and $\bar{X}_m^{i,\tau}$ $(m \ge 0)$ are defined as canonical $\mathbb{Z}^2$-valued Markov chains on their path spaces under the measure P^ω determined by the conditions

$$P^\omega\{\bar{Z}_0^{i,\tau} = (i, \tau)\} = 1,$$
$$P^\omega\{\bar{Z}_{s+1}^{i,\tau} = (y, \tau + s + 1) \mid \bar{Z}_s^{i,\tau} = (x, \tau + s)\} = u_{\tau+s}(x, y - x)$$

for the forward walk, and by

$$P^\omega\{\bar{X}_0^{i,\tau} = (i, \tau)\} = 1,$$
$$P^\omega\{\bar{X}_{s+1}^{i,\tau} = (y, \tau - s - 1) \mid \bar{X}_s^{i,\tau} = (x, \tau - s)\} = u_{\tau-s}(x, y - x)$$

for the backward walk. By dropping the time components τ, $\tau \pm s$ and $\tau \pm s \pm 1$ from the equations we get the corresponding properties for the spatial walks $Z_s^{i,\tau}$ and $X_s^{i,\tau}$. When we consider many walks under a common environment ω, it will be notationally convenient to attach the initial point (i, τ) to the walk and only the environment ω to the measure P^ω.

P^ω is called the *quenched* distribution, and expectation under P^ω is denoted by E^ω. The *annealed* distribution and expectation are $P(\cdot) = \mathbb{E}P^\omega(\cdot)$ and $E(\cdot) = \mathbb{E}E^\omega(\cdot)$. Under P both $X_m^{i,\tau}$ and $Z_m^{i,\tau}$ are ordinary homogeneous random walks on $\mathbb{Z}$ with jump probabilities $p(i, i + j) = p(0, j) = \mathbb{E}u_0(0, j)$. These walks satisfy the law of large numbers with velocity

$$V = \sum_{j \in \mathbb{Z}} p(0, j) j. \tag{3.3}$$

As for RAP, we also use the notation $b = -V$.

3.2. Limits for quenched mean processes. We start by stating the quenched invariance principle for the space-time RWRE. $\{B(t) : t \geq 0\}$ denotes standard one-dimensional Brownian motion. $D_{\mathbb{R}}[0, \infty)$ is the space of real-valued cadlag functions on $[0, \infty)$ with the standard Skorohod metric [12]. Recall the definition (2.5) of the variance σ_a^2 of the annealed walk, and assumption (2.2) that guarantees that the quenched walk has stochastic noise.

Theorem 3.1 [24]. *Assume* (2.2). *We have these bounds on the variance of the quenched mean: there exist constants C_1, C_2 such that for all n,*

$$C_1 n^{1/2} \leq \mathbb{E}\big[(E^\omega(X_n^{0,0}) - nV)^2\big] \leq C_2 n^{1/2}. \tag{3.4}$$

For $\mathbb{P}$-almost every ω, under P^ω the process $n^{-1/2}(X_{\lfloor nt \rfloor}^{0,0} - ntV)$ converges weakly to the process $B(\sigma_a^2 t)$ on the path space $D_{\mathbb{R}}[0, \infty)$ as $n \to \infty$.

Quite obviously, $X_n^{0,0}$ and $Z_n^{0,0}$ are interchangeable in the above theorem. Bounds (3.4) suggest the possibility of a weak limit for the quenched mean on the scale $n^{1/4}$. Such results are the main point of this section.

For $t \geq 0$, $r \in \mathbb{R}$ we define scaled, centered quenched mean processes

$$a_n(t, r) = n^{-1/4} \Big\{ E^\omega\big(Z_{\lfloor nt \rfloor}^{\lfloor r\sqrt{n} \rfloor, 0}\big) - \lfloor r\sqrt{n} \rfloor - \lfloor nt \rfloor V \Big\} \tag{3.5}$$

for the forward walks, and

$$y_n(t, r) = n^{-1/4} \Big\{ E^\omega\big(X_{\lfloor nt \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n} \rfloor, \lfloor nt \rfloor}\big) - \lfloor r\sqrt{n} \rfloor \Big\} \tag{3.6}$$

for the backward walks. In words, the process a_n follows forward walks from level 0 to level $\lfloor nt \rfloor$ and records centered quenched means. Process y_n follows backward walks from level $\lfloor nt \rfloor$ down to level 0 and records the centered quenched mean of the point it hits at level 0. The initial points of the backward walks are translated by the negative of the mean drift $\lfloor ntb \rfloor$. This way the temporal processes $a_n(\cdot, r)$ and $y_n(\cdot, r)$ obtained by fixing r are meaningful processes.

Random variable $y_n(t, r)$ is not exactly centered, for

$$\mathbb{E} y_n(t, r) = n^{-1/4}\big(\lfloor ntb \rfloor - \lfloor nt \rfloor b\big). \tag{3.7}$$

Of course this makes no difference to the limit.

Next we describe the Gaussian limiting processes. Recall the constant κ defined in (2.14) and the function Γ_q defined in (2.15). Let $\{a(t, r) : (t, r) \in \mathbb{R}_+ \times \mathbb{R}\}$ and $\{y(t, r) : (t, r) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mean zero Gaussian processes with covariances

$$Ea(s, q)a(t, r) = \Gamma_q\big((s \wedge t, q), (s \wedge t, r)\big)$$

and

$$Ey(s, q)y(t, r) = \Gamma_q\big((s, q), (t, r)\big)$$

for $s, t \geq 0$ and $q, r \in \mathbb{R}$. When one argument is fixed, the random function $r \mapsto y(t, r)$ is denoted by $y(t, \cdot)$ and $t \mapsto y(t, r)$ by $y(\cdot, r)$. From the covariances follows that at a fixed time level t the spatial processes $a(t, \cdot)$ and $y(t, \cdot)$ are equal in distribution.

We record basic properties of these processes.

Lemma 3.1. *The process* $\{y(t,r)\}$ *has a version with continuous paths as functions of* (t,r)*. Furthermore, it has the following Markovian structure in time. Given* $0 = t_0 < t_1 < \cdots < t_n$*, let* $\{\tilde{y}(t_i - t_{i-1}, \cdot) : 1 \le i \le n\}$ *be independent random functions such that* $\tilde{y}(t_i - t_{i-1}, \cdot)$ *has the distribution of* $y(t_i - t_{i-1}, \cdot)$ *for* $i = 1, \dots, n$*. Define* $y^*(t_1, r) = \tilde{y}(t_1, r)$ *for* $r \in \mathbb{R}$*, and then inductively for* $i = 2, \dots, n$ *and* $r \in \mathbb{R}$*,*

$$y^*(t_i, r) = \int_{\mathbb{R}} \varphi_{\sigma_a^2(t_i - t_{i-1})}(u) y^*(t_{i-1}, r+u)\, du + \tilde{y}(t_i - t_{i-1}, r). \tag{3.8}$$

Then the joint distribution of the random functions $\{y^*(t_i, \cdot) : 1 \le i \le n\}$ *is the same as that of* $\{y(t_i, \cdot) : 1 \le i \le n\}$ *from the original process.*

Sketch of proof. Consider (s,q) and (t,r) varying in a compact set. From the covariance comes the estimate

$$E\big[(y(s,q) - y(t,r))^2\big] \le C\big(|s-t|^{1/2} + |q-r|\big) \tag{3.9}$$

from which, since the integrand is Gaussian,

$$E\big[(y(s,q) - y(t,r))^{10}\big] \le C\big(|s-t|^{1/2} + |q-r|\big)^5 \le C\, \|(s,q) - (t,r)\|^{5/2}. \tag{3.10}$$

Kolmogorov's criterion implies the existence of a continuous version.

For the second statement use (3.8) to express a linear combination $\sum_{i=1}^n \theta_i y^*(t_i, r_i)$ in the form

$$\sum_{i=1}^n \theta_i y^*(t_i, r_i) = \sum_{i=1}^n \int_{\mathbb{R}} \tilde{y}(t_i - t_{i-1}, x)\, \lambda_i(dx),$$

where the signed measures λ_i are linear combinations of Gaussian distributions. Use this representation to compute the variance of the linear combination on the left-hand side (it is mean zero Gaussian). Observe that this variance equals

$$\sum_{i,j} \theta_i \theta_j \Gamma_q((t_i, r_i), (t_j, r_j)).$$

□

Lemma 3.2. *The process* $\{a(t,r)\}$ *has a version with continuous paths as functions of* (t,r)*. Furthermore, it has independent increments in time. A more precise statement follows. Given* $0 = t_0 < t_1 < \cdots < t_n$*, let* $\{\tilde{a}(t_i - t_{i-1}, \cdot) : 1 \le i \le n\}$ *be independent random functions such that* $\tilde{a}(t_i - t_{i-1}, \cdot)$ *has the distribution of* $a(t_i - t_{i-1}, \cdot)$ *for* $i = 1, \dots, n$*. Define* $a^*(t_1, r) = \tilde{a}(t_1, r)$ *for* $r \in \mathbb{R}$*, and then inductively for* $i = 2, \dots, n$ *and* $r \in \mathbb{R}$*,*

$$a^*(t_i, r) = a^*(t_{i-1}, r) + \int_{\mathbb{R}} \varphi_{\sigma_a^2 t_{i-1}}(u) \tilde{a}(t_i - t_{i-1}, r+u)\, du. \tag{3.11}$$

Then the joint distribution of the random functions $\{a^*(t_i, \cdot) : 1 \le i \le n\}$ *is the same as that of* $\{a(t_i, \cdot) : 1 \le i \le n\}$ *from the original process.*

The proof of the lemma above is similar to the previous one so we omit it.

Remark 3.1. Processes y and a have representations in terms of stochastic integrals.

As in Remark 2.1 let W be a two-parameter Brownian motion on $\mathbb{R}_+ \times \mathbb{R}$. In more technical terms, W is the orthogonal Gaussian martingale measure on $\mathbb{R}_+ \times \mathbb{R}$ with covariance $EW([0,s] \times A)W([0,t] \times B) = (s \wedge t)\,\mathrm{Leb}(A \cap B)$ for $s, t \in \mathbb{R}_+$ and bounded Borel sets $A, B \subseteq \mathbb{R}$. Then

$$y(t,r) = \sigma_a \sqrt{\kappa} \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma_a^2(t-s)}(r-z)\, dW(s,z) \tag{3.12}$$

while

$$a(t,r) = \sigma_a \sqrt{\kappa} \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma_a^2 s}(r-z)\, dW(s,z). \tag{3.13}$$

By the equations above we mean equality in distribution of processes. They can be verified by a comparison of covariances, as the integrals on the right are also Gaussian processes. Formula (3.12) implies that process $\{y(t,r)\}$ is a weak solution of the stochastic heat equation

$$y_t = \tfrac{1}{2}\sigma_a^2\, y_{rr} + \sigma_a \sqrt{\kappa}\, \dot{W}\,, \qquad y(0,r) \equiv 0, \tag{3.14}$$

where $\dot{W}$ is white noise. (See [34].) These observations are not used elsewhere in the paper.

Next we record the limits for the quenched mean processes. The four theorems that follow require assumption (2.2) of stochastic noise and the assumption that the annealed probabilities $p(0,j) = \mathbb{E}u_0^\omega(0,j)$ have span 1. This next theorem is the one needed for Theorem 2.1 for RAP.

Theorem 3.2. *The finite dimensional distributions of processes $y_n(t,r)$ converge to those of $y(t,r)$ as $n \to \infty$. More precisely, for any finite set of points $\{(t_j, r_j) : 1 \le j \le k\}$ in $\mathbb{R}_+ \times \mathbb{R}$, the vector $\big(y_n(t_j, r_j) : 1 \le j \le k\big)$ converges weakly in $\mathbb{R}^k$ to the vector $\big(y(t_j, r_j) : 1 \le j \le k\big)$.*

Observe that property (3.8) is easy to understand from the limit. It reflects the Markovian property

$$E^\omega(X_\tau^{x,\tau}) = \sum_y P^\omega(X_{\tau-s}^{x,\tau} = y)E^\omega(X_s^{y,s}) \quad \text{for } s < \tau,$$

and the "homogenization" of the coefficients which converge to Gaussian probabilities by the quenched central limit theorem.

Let us restrict the backward quenched mean process to a single characteristic to observe the outcome. This is the source of the first term in the temporal correlations (2.19) for RAP. The next statement needs no proof, for it is just a particular case of the limit in Theorem 3.2.

Corollary 3.3. *Fix $r \in \mathbb{R}$. As $n \to \infty$, the finite dimensional distributions of the process $\{y_n(t,r) : t \ge 0\}$ converge to those of the mean zero Gaussian process $\{y(t) : t \ge 0\}$ with covariance*

$$Ey(s)y(t) = \frac{\kappa\sigma_a}{\sqrt{2\pi}}\big(\sqrt{t+s} - \sqrt{t-s}\big) \qquad (s < t).$$

Then the same for the forward processes.

Theorem 3.4. *The finite dimensional distributions of processes a_n converge to those of a as $n \to \infty$. More precisely, for any finite set of points $\{(t_j, r_j) : 1 \le j \le k\}$ in $\mathbb{R}_+ \times \mathbb{R}$, the vector $\big(a_n(t_j, r_j) : 1 \le j \le k\big)$ converges weakly in $\mathbb{R}^k$ to the vector $\big(a(t_j, r_j) : 1 \le j \le k\big)$.*

When we specialize to a temporal process we also verify path-level tightness and hence get weak convergence of the entire process. When $r = q$ in (2.16) we get

$$\Gamma_q\big((s \wedge t, r), (s \wedge t, r)\big) = c_a \sqrt{s \wedge t}$$

with $c_a = \sigma_D^2/(\beta\sqrt{\pi\sigma_a^2})$. Since $s \wedge t$ is the covariance of standard Brownian motion $B(\cdot)$, we get the following limit.

Corollary 3.5. *Fix $r \in \mathbb{R}$. As $n \to \infty$, the process $\{a_n(t, r) : t \ge 0\}$ converges weakly to $\{B(c_a\sqrt{t}\,) : t \ge 0\}$ on the path space $D_{\mathbb{R}}[0, \infty)$.*

4. Random Walk Preliminaries

In this section we collect some auxiliary results for random walks. The basic assumptions, (2.2) and span 1 for the $p(0, j) = \mathbb{E}u_0(0, j)$ walk, are in force throughout the remainder of the paper.

Recall the drift in the e_1 direction at the origin defined by

$$D(\omega) = \sum_{x \in \mathbb{Z}} x\ u_0^{\omega}(0, x),$$

with mean $V = -b = \mathbb{E}(D)$. Define the centered drift by

$$g(\omega) = D(\omega) - V = E^{\omega}(X_1^{0,0} - V).$$

The variance is $\sigma_D^2 = \mathbb{E}[g^2]$. The variance of the i.i.d. annealed walk in the e_1 direction is

$$\sigma_a^2 = \sum_{x \in \mathbb{Z}} (x - V)^2\ \mathbb{E}u_0^{\omega}(0, x).$$

These variances are connected by

$$\sigma_a^2 = \sigma_D^2 + E\left[(X_1^{0,0} - D)^2\right].$$

Let X_n and $\tilde{X}_n$ be two independent walks in a common environment ω, and $Y_n = X_n - \tilde{X}_n$. In the annealed sense Y_n is a Markov chain on $\mathbb{Z}$ with transition probabilities

$$q(0, y) = \sum_{z \in \mathbb{Z}} \mathbb{E}[u_0(0, z)u_0(0, z + y)] \qquad (y \in \mathbb{Z}),$$
$$q(x, y) = \sum_{z \in \mathbb{Z}} p(0, z)p(0, z + y - x) \qquad (x \ne 0, y \in \mathbb{Z}).$$

Y_n can be thought of as a symmetric random walk on $\mathbb{Z}$ whose transition has been perturbed at the origin. The corresponding homogeneous, unperturbed transition probabilities are

$$\bar{q}(x,y) = \bar{q}(0,y-x) = \sum_{z\in\mathbb{Z}} p(0,z)p(0,z+y-x) \qquad (x,y\in\mathbb{Z}).$$

The $\bar{q}$-walk has variance $2\sigma_a^2$ and span 1 as can be deduced from the definition and the hypothesis that the p-walk has span 1. Since the $\bar{q}$-walk is symmetric, its range must be a subgroup of $\mathbb{Z}$. Then span 1 implies that it is irreducible. The $\bar{q}$-walk is recurrent by the Chung-Fuchs theorem. Elementary arguments extend irreducibility and recurrence from $\bar{q}$ to the q-chain because away from the origin the two walks are the same. Note that assumption (2.2) is required here because the q-walk is absorbed at the origin iff (2.2) fails.

Note that the functions defined in (2.7) are the characteristic functions of these transitions:

$$\lambda(t) = \sum_x q(0,x)e^{itx} \quad \text{and} \quad \bar{\lambda}(t) = \sum_x \bar{q}(0,x)e^{itx}.$$

Multistep transitions are denoted by $q^k(x,y)$ and $\bar{q}^k(x,y)$, defined as usual by

$$\bar{q}^0(x,y) = \mathbf{1}_{\{x=y\}}, \quad \bar{q}^1(x,y) = q(x,y),$$
$$\bar{q}^k(x,y) = \sum_{x_1,\dots,x_{k-1}\in\mathbb{Z}} \bar{q}(x,x_1)\bar{q}(x_1,x_2)\cdots\bar{q}(x_{k-1},y) \quad (k\geq 2).$$

Green functions for the $\bar{q}$- and q-walks are

$$\bar{G}_n(x,y) = \sum_{k=0}^{n} \bar{q}^k(x,y) \quad \text{and} \quad G_n(x,y) = \sum_{k=0}^{n} q^k(x,y).$$

$\bar{G}_n$ is symmetric but G_n not necessarily.

The potential kernel $\bar{a}$ of the $\bar{q}$-walk is defined by

$$\bar{a}(x) = \lim_{n\to\infty} \left\{\bar{G}_n(0,0) - \bar{G}_n(x,0)\right\}. \tag{4.1}$$

It satisfies $\bar{a}(0) = 0$, the equations

$$\bar{a}(x) = \sum_{y\in\mathbb{Z}} \bar{q}(x,y)\bar{a}(y) \quad \text{for } x\neq 0, \text{ and} \quad \sum_{y\in\mathbb{Z}} \bar{q}(0,y)\bar{a}(y) = 1, \tag{4.2}$$

and the limit

$$\lim_{x\to\pm\infty} \frac{\bar{a}(x)}{|x|} = \frac{1}{2\sigma_a^2}. \tag{4.3}$$

These facts can be found in Sects. 28 and 29 of Spitzer's monograph [31].

Example 4.1. If for some $k\in\mathbb{Z}$, $p(0,k) + p(0,k+1) = 1$, so that $\bar{q}(0,x) = 0$ for $x\notin\{-1,0,1\}$, then $\bar{a}(x) = |x|/(2\sigma_a^2)$.

Define the constant

$$\beta = \sum_{x \in \mathbb{Z}} q(0,x)\bar{a}(x). \tag{4.4}$$

To see that this definition agrees with (2.8), observe that the above equality leads to

$$\beta = \lim_{n\to\infty} \left\{ \sum_{k=0}^{n} \bar{q}^k(0,0) - \sum_{k=0}^{n}\sum_{x} q(0,x)\bar{q}^k(x,0) \right\}.$$

Think of the last sum over x as $P[Y_1 + \bar{Y}_k = 0]$, where $\bar{Y}_k$ is the $\bar{q}$-walk, and Y_1 and $\bar{Y}_k$ are independent. Since $Y_1 + \bar{Y}_k$ has characteristic function $\lambda(t)\bar{\lambda}^k(t)$, we get

$$\beta = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-\lambda(t)) \sum_{k=0}^{n} \bar{\lambda}^k(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\lambda(t)}{1-\bar{\lambda}(t)}\, dt.$$

Ferrari and Fontes [14] begin their development by showing that

$$\beta = \lim_{s\nearrow 1} \frac{\bar{\zeta}(s)}{\zeta(s)},$$

where ζ and $\bar{\zeta}$ are the generating functions

$$\zeta(s) = \sum_{k=0}^{\infty} q^k(0,0)s^k \quad \text{and} \quad \bar{\zeta}(s) = \sum_{k=0}^{\infty} \bar{q}^k(0,0)s^k.$$

Our development bypasses the generating functions. We begin with the asymptotics of the Green functions. This is the key to all our results, both for RWRE and RAP. As already pointed out, without assumption (2.2) the result would be completely wrong because the q-walk absorbs at 0, while a span $h > 1$ would appear in this limit as an extra factor.

Lemma 4.1. *Let $x \in \mathbb{R}$, and let x_n be any sequence of integers such that $x_n - n^{1/2}x$ stays bounded. Then*

$$\lim_{n\to\infty} n^{-1/2} G_n\big(x_n, 0\big) = \frac{1}{2\beta\sigma_a^2}\int_0^{2\sigma_a^2} \frac{1}{\sqrt{2\pi v}} \exp\Big\{-\frac{x^2}{2v}\Big\}\, dv. \tag{4.5}$$

Proof. For the homogeneous $\bar{q}$-walk the local limit theorem [11, Sect. 2.5] implies that

$$\lim_{n\to\infty} n^{-1/2} \bar{G}_n(0, x_n) = \frac{1}{2\sigma_a^2}\int_0^{2\sigma_a^2} \frac{1}{\sqrt{2\pi v}} \exp\Big\{-\frac{x^2}{2v}\Big\}\, dv \tag{4.6}$$

and by symmetry the same limit is true for $n^{-1/2}\bar{G}_n(x_n, 0)$. In particular,

$$\lim_{n\to\infty} n^{-1/2} \bar{G}_n(0,0) = \frac{1}{\sqrt{\pi\sigma_a^2}}. \tag{4.7}$$

Next we show

$$\lim_{n\to\infty} n^{-1/2} G_n(0,0) = \frac{1}{\beta\sqrt{\pi\sigma_a^2}}. \tag{4.8}$$

Using (4.2), $\bar{a}(0) = 0$, and $\bar{q}(x, y) = q(x, y)$ for $x \neq 0$ we develop

$$\begin{aligned}\sum_{x\in\mathbb{Z}} q^m(0,x)\bar{a}(x) &= \sum_{x\neq 0} q^m(0,x)\bar{a}(x) = \sum_{x\neq 0, y\in\mathbb{Z}} q^m(0,x)\bar{q}(x,y)\bar{a}(y)\\ &= \sum_{x\neq 0, y\in\mathbb{Z}} q^m(0,x)q(x,y)\bar{a}(y)\\ &= \sum_{y\in\mathbb{Z}} q^{m+1}(0,y)\bar{a}(y) - q^m(0,0)\sum_{y\in\mathbb{Z}} q(0,y)\bar{a}(y).\end{aligned}$$

Identify β in the last sum above and sum over $m = 0, 1, \ldots, n-1$ to get

$$\big(1 + q(0,0) + \cdots + q^{n-1}(0,0)\big)\beta = \sum_{x\in\mathbb{Z}} q^n(0,x)\bar{a}(x).$$

Write this in the form

$$n^{-1/2}G_{n-1}(0,0)\beta = n^{-1/2}E_0\big[\bar{a}(Y_n)\big].$$

Recall that $Y_n = X_n - \widetilde{X}_n$, where X_n and $\widetilde{X}_n$ are two independent walks in the same environment. Thus by Theorem 3.1 $n^{-1/2}Y_n$ converges weakly to a centered Gaussian with variance $2\sigma_a^2$. Under the annealed measure the walks X_n and $\widetilde{X}_n$ are ordinary i.i.d. walks with bounded steps, hence there is enough uniform integrability to conclude that $n^{-1/2}E_0\,|Y_n| \to 2\sqrt{\sigma_a^2/\pi}$. By (4.3) and straightforward estimation,

$$n^{-1/2}E_0\big[\bar{a}(Y_n)\big] \to \frac{1}{\sqrt{\sigma_a^2\pi}}\,.$$

This proves (4.8).

From (4.7)–(4.8) we take the conclusion

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\big|\beta G_n(0,0) - \bar{G}_n(0,0)\big| = 0. \tag{4.9}$$

Let $f^0(z, 0) = \mathbf{1}_{\{z=0\}}$ and for $k \geq 1$ let

$$f^k(z,0) = \mathbf{1}_{\{z\neq 0\}} \sum_{z_1\neq 0,\ldots,z_{k-1}\neq 0} q(z,z_1)q(z_1,z_2)\cdots q(z_{k-1},0).$$

This is the probability that the first visit to the origin occurs at time k, including a possible first visit at time 0. Note that this quantity is the same for the q and $\bar{q}$ walks. Now bound

$$\begin{aligned}&\sup_{z\in\mathbb{Z}} \Big|\frac{\beta}{\sqrt{n}}G_n(z,0) - \frac{1}{\sqrt{n}}\bar{G}_n(z,0)\Big|\\ &\quad\leq \sup_{z\in\mathbb{Z}} \frac{1}{\sqrt{n}}\sum_{k=0}^{n} f^k(z,0)\big|\beta G_{n-k}(0,0) - \bar{G}_{n-k}(0,0)\big|.\end{aligned}$$

To see that the last line vanishes as $n \to \infty$, by (4.9) choose n_0 so that

$$|\beta G_{n-k}(0,0) - \bar{G}_{n-k}(0,0)| \leq \varepsilon\sqrt{n-k}$$

for $k \leq n - n_0$, while trivially

$$|\beta G_{n-k}(0,0) - \bar{G}_{n-k}(0,0)| \leq Cn_0$$

for $n - n_0 < k \leq n$. The conclusion (4.5) now follows from this and (4.6). $\square$

Lemma 4.2. $\sup_{n\geq 1}\sup_{x\in\mathbb{Z}}\big|G_n(x,0)-G_n(x+1,0)\big|<\infty$.

Proof. Let $T_y=\inf\{n\geq 1: Y_n=y\}$ denote the first hitting time of the point y,

$$\begin{aligned}G_n(x,0)&=E_x\Big[\sum_{k=0}^{n}\mathbf{1}\{Y_k=0\}\Big]=E_x\Big[\sum_{k=0}^{T_y\wedge n}\mathbf{1}\{Y_k=0\}\Big]+E_x\Big[\sum_{k=T_y\wedge n+1}^{n}\mathbf{1}\{Y_k=0\}\Big]\\&\leq E_x\Big[\sum_{k=0}^{T_y}\mathbf{1}\{Y_k=0\}\Big]+G_n(y,0).\end{aligned}$$

In an irreducible Markov chain the expectation $E_x\big[\sum_{k=0}^{T_y}\mathbf{1}\{Y_k=0\}\big]$ is finite for any given states x,y [8, Theorem 3 in Sect. I.9]. Since this is independent of n, the inequalities above show that

$$\sup_n\sup_{-a\leq x\leq a}|G_n(x,0)-G_n(x+1,0)|<\infty \tag{4.10}$$

for any fixed a.

Fix a positive integer a larger than the range of the jump kernels $q(x,y)$ and $\bar{q}(x,y)$. Consider $x>a$. Let $\sigma=\inf\{n\geq 1: Y_n\leq a-1\}$ and $\tau=\inf\{n\geq 1: Y_n\leq a\}$. Since the q-walks starting at x and $x+1$ obey the translation-invariant kernel $\bar{q}$ until they hit the origin,

$$P_x[Y_\sigma=y,\sigma=n]=P_{x+1}[Y_\tau=y+1,\tau=n].$$

(Any path that starts at x and enters $[0,a-1]$ at y can be translated by 1 to a path that starts at $x+1$ and enters $[0,a]$ at $y+1$, without changing its probability.) Consequently

$$\begin{aligned}&G_n(x,0)-G_n(x+1,0)\\&\quad=\sum_{k=1}^{n}\sum_{y=0}^{a-1}P_x[Y_\sigma=y,\sigma=k]\big(G_{n-k}(y,0)-G_{n-k}(y+1,0)\big).\end{aligned}$$

Together with (4.10) this shows that the quantity in the statement of the lemma is uniformly bounded over $x\geq 0$. The same argument works for $x\leq 0$. $\square$

One can also derive the limit

$$\lim_{n\to\infty}\big\{G_n(0,0)-G_n(x,0)\big\}=\beta^{-1}\bar{a}(x)$$

but we have no need for this.

Lastly, a moderate deviation bound for the space-time RWRE with bounded steps. Let $X_s^{i,\tau}$ be the spatial backward walk defined in Sect. 3 with the bound (3.2) on the steps. Let $\widetilde{X}_s^{i,\tau}=X_s^{i,\tau}-i-Vs$ be the centered walk.

Lemma 4.3. *For $m,n\in\mathbb{N}$, let $(i(m,n),\tau(m,n))\in\mathbb{Z}^2$, $v(n)\geq 1$, and let $s(n)\to\infty$ be a sequence of positive integers. Let α, γ and c be positive reals. Assume*

$$\sum_{n=1}^{\infty}v(n)s(n)^\alpha\exp\{-cs(n)^\gamma\}<\infty.$$

Then for $\mathbb{P}$*-almost every* ω,

$$\lim_{n\to\infty} \max_{1\le m\le v(n)} s(n)^\alpha P^\omega\Big\{\max_{1\le k\le s(n)} \widetilde{X}_k^{i(m,n),\,\tau(m,n)} \ge cs(n)^{\frac12+\gamma}\Big\} = 0. \tag{4.11}$$

Proof. Fix $\varepsilon > 0$. By Markov's inequality and translation-invariance,

$$\mathbb{P}\Big[\omega : \max_{1\le m\le v(n)} s(n)^\alpha P^\omega\Big\{\max_{1\le k\le s(n)} \widetilde{X}_k^{i(m,n),\,\tau(m,n)} \ge cs(n)^{\frac12+\gamma}\Big\} \ge \varepsilon\Big]$$
$$\le \varepsilon^{-1}s(n)^\alpha v(n) P\left\{\max_{1\le k\le s(n)} \widetilde{X}_k^{0,0} \ge cs(n)^{\frac12+\gamma}\right\}.$$

Under the annealed measure P, $\widetilde{X}_k^{0,0}$ is an ordinary homogeneous mean zero random walk with bounded steps. It has a finite moment generating function $\phi(\lambda) = \log E(\exp\{\lambda\widetilde{X}_1^{0,0}\})$ that satisfies $\phi(\lambda) = \mathcal{O}(\lambda^2)$ for small λ. Apply Doob's inequality to the martingale $M_k = \exp(\lambda\widetilde{X}_k^{0,0} - k\phi(\lambda))$, note that $\phi(\lambda) \ge 0$, and choose a constant a_1 such that $\phi(\lambda) \le a_1\lambda^2$ for small λ. This gives

$$P\left\{\max_{1\le k\le s(n)} \widetilde{X}_k^{0,0} \ge cs(n)^{\frac12+\gamma}\right\} \le P\left\{\max_{1\le k\le s(n)} M_k \ge \exp\big(c\lambda s(n)^{\frac12+\gamma} - s(n)\phi(\lambda)\big)\right\}$$
$$\le \exp\big(-c\lambda s(n)^{\frac12+\gamma} + a_1 s(n)\lambda^2\big) = e^{a_1}\cdot\exp\{-cs(n)^\gamma\},$$

where we took $\lambda = s(n)^{-\frac12}$.

The conclusion of the lemma now follows from the hypothesis and Borel-Cantelli. □

5. Proofs for Backward Walks in a Random Environment

Here are two further notational conventions used in the proofs. The environment configuration at a fixed time level is denoted by $\bar\omega_n = \{\omega_{x,n} : x \in \mathbb{Z}\}$. Translations on Ω are defined by $(T_{x,n}\omega)_{y,k} = \omega_{x+y,n+k}$.

5.1. Proof of Theorem 3.2. This proof proceeds in two stages. First in Lemma 5.1 convergence is proved for finite-dimensional distributions at a fixed t-level. In the second stage the convergence is extended to multiple t-levels via the natural Markovian property that we express in terms of y_n next. Abbreviate $X_k^{n,t,r} = X_k^{\lfloor ntb\rfloor+\lfloor r\sqrt{n}\rfloor,\,\lfloor nt\rfloor}$. Then for $0 \le s < t$,

$$\begin{aligned} y_n(t,r) &= n^{-1/4}\big(E^\omega(X_{\lfloor nt\rfloor}^{n,t,r}) - \lfloor r\sqrt{n}\rfloor\big) \\ &= \sum_{z\in\mathbb{Z}} P^\omega\big\{X_{\lfloor nt\rfloor-\lfloor ns\rfloor}^{n,t,r} = \lfloor nsb\rfloor + z\big\} n^{-1/4}\big(E^\omega(X_{\lfloor ns\rfloor}^{\lfloor nsb\rfloor+z,\,\lfloor ns\rfloor}) - z\big) \\ &\quad + \sum_{z\in\mathbb{Z}} P^\omega\big\{X_{\lfloor nt\rfloor-\lfloor ns\rfloor}^{n,t,r} = \lfloor nsb\rfloor + z\big\} n^{-1/4}\big(z - \lfloor r\sqrt{n}\rfloor\big) \\ &= \sum_{z\in\mathbb{Z}} P^\omega\big\{X_{\lfloor nt\rfloor-\lfloor ns\rfloor}^{n,t,r} = \lfloor nsb\rfloor + z\big\} n^{-1/4}\big(E^\omega(X_{\lfloor ns\rfloor}^{\lfloor nsb\rfloor+z,\,\lfloor ns\rfloor}) - z\big) \\ &\quad + n^{-1/4}\big\{E^\omega\big(X_{\lfloor nt\rfloor-\lfloor ns\rfloor}^{n,t,r}\big) - \lfloor nsb\rfloor - \lfloor r\sqrt{n}\rfloor\big\} \end{aligned}$$

$$= \sum_{z\in\mathbb{Z}} P^\omega\big\{X^{n,t,r}_{\lfloor nt\rfloor-\lfloor ns\rfloor} = \lfloor nsb\rfloor + z\big\} n^{-1/4}\big(E^\omega(X^{\lfloor nsb\rfloor+z,\lfloor ns\rfloor}_{\lfloor ns\rfloor}) - z\big) \tag{5.1}$$

$$+y_n(u_n, r)\circ T_{\lfloor ntb\rfloor-\lfloor nbu_n\rfloor,\lfloor nt\rfloor-\lfloor nu_n\rfloor} + n^{-1/4}\big(\lfloor ntb\rfloor - \lfloor nsb\rfloor - \lfloor nbu_n\rfloor\big), \tag{5.2}$$

where we defined $u_n = n^{-1}(\lfloor nt\rfloor - \lfloor ns\rfloor)$ so that $\lfloor nu_n\rfloor = \lfloor nt\rfloor - \lfloor ns\rfloor$. $T_{x,m}$ denotes the translation of the random environment that makes (x, m) the new space-time origin, in other words $(T_{x,m}\omega)_{y,n} = \omega_{x+y,m+n}$.

The key to making use of the decomposition of $y_n(t, r)$ given on lines (5.1) and (5.2) is that the quenched expectations

$$E^\omega\left(X^{\lfloor nsb\rfloor+z,\lfloor ns\rfloor}_{\lfloor ns\rfloor}\right) \quad \text{and} \quad y_n(u_n, r)\circ T_{\lfloor ntb\rfloor-\lfloor nbu_n\rfloor,\lfloor nt\rfloor-\lfloor nu_n\rfloor}$$

are independent because they are functions of environments $\bar\omega_m$ on disjoint sets of levels m, while the coefficients $P^\omega\big\{X^{n,t,r}_{\lfloor nt\rfloor-\lfloor ns\rfloor} = \lfloor nsb\rfloor + z\big\}$ on line (5.1) converge (in probability) to Gaussian probabilities by the quenched CLT as $n \to \infty$. In the limit this decomposition becomes (3.8).

Because of the little technicality of matching $\lfloor nt\rfloor - \lfloor ns\rfloor$ with $\lfloor n(t - s)\rfloor$ we state the next lemma for a sequence $t_n \to t$ instead of a fixed t.

Lemma 5.1. *Fix $t > 0$, and finitely many reals $r_1 < r_2 < \ldots < r_N$. Let t_n be a sequence of positive reals such that $t_n \to t$. Then as $n \to \infty$ the $\mathbb{R}^N$-valued vector $(y_n(t_n, r_1), \ldots, y_n(t_n, r_N))$ converges weakly to a mean zero Gaussian vector with covariance matrix $\{\Gamma_q((t, r_i), (t, r_j)) : 1 \le i, j \le N\}$ with Γ_q as defined in* (2.15).

The proof of Lemma 5.1 is technical (martingale CLT and random walk estimates), so we postpone it and proceed with the main development.

Proof of Theorem 3.2. The argument is inductive on the number M of time points in the finite-dimensional distribution. The induction assumption is that

$$\begin{aligned}&[y_n(t_i, r_j) : 1 \le i \le M, 1 \le j \le N] \to [y(t_i, r_j) : 1 \le i \le M, 1 \le j \le N]\\ &\text{weakly on } \mathbb{R}^{MN} \text{ for any } M \text{ time points } 0 \le t_1 < t_2 < \cdots < t_M \text{ and}\\ &\text{for any reals } r_1, \ldots, r_N \text{ for any finite } N.\end{aligned} \tag{5.3}$$

The case $M = 1$ comes from Lemma 5.1. To handle the case $M + 1$, let $0 \le t_1 < t_2 < \cdots < t_{M+1}$, and fix an arbitrary $(M + 1)N$-vector $[\theta_{i,j}]$. By the Cramér-Wold device, it suffices to show the weak convergence of the linear combination

$$\sum_{\substack{1\le i\le M+1\\1\le j\le N}} \theta_{i,j}\, y_n(t_i, r_j) = \sum_{\substack{1\le i\le M\\1\le j\le N}} \theta_{i,j}\, y_n(t_i, r_j) + \sum_{1\le j\le N} \theta_{M+1,j}\, y_n(t_{M+1}, r_j), \tag{5.4}$$

where we separated out the $(M + 1)$-term to be manipulated. The argument will use (5.1)–(5.2) to replace the values at t_{M+1} with values at t_M plus terms independent of the rest.

For Borel sets $B \subseteq \mathbb{R}$ define the probability measure

$$p^\omega_{n,j}(B) = P^\omega\big\{X^{\lfloor nt_{M+1}b\rfloor+\lfloor r_j\sqrt{n}\rfloor,\lfloor nt_{M+1}\rfloor}_{\lfloor nt_{M+1}\rfloor-\lfloor nt_M\rfloor} - \lfloor nt_M b\rfloor \in B\big\}.$$

Apply the decomposition (5.1)–(5.2), with $s_n = n^{-1}(\lfloor nt_{M+1}\rfloor - \lfloor nt_M\rfloor)$ and

$$\tilde{y}_n(s_n, r_j) = y_n(s_n, r_j) \circ T_{\lfloor nt_{M+1}b\rfloor - \lfloor ns_n b\rfloor, \lfloor nt_{M+1}\rfloor - \lfloor ns_n\rfloor}$$

to get

$$\begin{aligned} y_n(t_{M+1}, r_j) = \sum_{z\in\mathbb{Z}} p^{\omega}_{n,j}(z) n^{-1/4}\big\{E^{\omega}\Big(X^{\lfloor nt_M b\rfloor + z, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor}\Big) - z\big\} \\ + \tilde{y}_n(s_n, r_j) + \mathcal{O}(n^{-1/4}). \end{aligned} \tag{5.5}$$

The $\mathcal{O}(n^{-1/4})$ term above is $n^{-1/4}\big(\lfloor nt_{M+1}b\rfloor - \lfloor nt_M b\rfloor - \lfloor ns_n b\rfloor\big)$, a deterministic quantity. Next we reorganize the sum in (5.5) to take advantage of Lemma 5.1. Given $a > 0$, define a partition of $[-a, a]$ by

$$-a = u_0 < u_1 < \cdots < u_L = a$$

with mesh $\Delta = \max\{u_{\ell+1} - u_\ell\}$. For integers z such that $-a\sqrt{n} < z \le a\sqrt{n}$, let $u(z)$ denote the value u_ℓ such that $u_\ell\sqrt{n} < z \le u_{\ell+1}\sqrt{n}$. For $1 \le j \le N$ define an error term by

$$\begin{aligned} R_{n,j}(a) = n^{-1/4} \sum_{z=\lfloor -a\sqrt{n}\rfloor+1}^{\lfloor a\sqrt{n}\rfloor} & p^{\omega}_{n,j}(z)\Big(\big\{E^{\omega}(X^{\lfloor nt_M b\rfloor+z, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor}) - z\big\} \\ & - \big\{E^{\omega}(X^{\lfloor nt_M b\rfloor + \lfloor u(z)\sqrt{n}\rfloor, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor}) - \lfloor u(z)\sqrt{n}\rfloor\big\}\Big) \qquad (5.6) \\ & + n^{-1/4} \sum_{z\le -a\sqrt{n},\, z> a\sqrt{n}} p^{\omega}_{n,j}(z)\big\{E^{\omega}(X^{\lfloor nt_M b\rfloor+z, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor}) - z\big\}. \qquad (5.7) \end{aligned}$$

With this we can rewrite (5.5) as

$$\begin{aligned} y_n(t_{M+1}, r_j) = \sum_{\ell=0}^{L-1} p^{\omega}_{n,j}(u_\ell n^{1/2}, u_{\ell+1}n^{1/2}\,] y_n(t_M, u_\ell) + \tilde{y}_n(s_n, r_j) \\ + R_{n,j}(a) + \mathcal{O}(n^{-1/4}). \end{aligned} \tag{5.8}$$

Let γ denote a normal distribution on $\mathbb{R}$ with mean zero and variance $\sigma_a^2(t_{M+1} - t_M)$. According to the quenched CLT Theorem 3.1,

$$p^{\omega}_{n,j}(u_\ell n^{1/2}, u_{\ell+1}n^{1/2}\,] \to \gamma(u_\ell - r_j, u_{\ell+1} - r_j] \quad \text{in } \mathbb{P}\text{-probability as } n \to \infty. \tag{5.9}$$

In view of (5.4) and (5.8), we can write

$$\begin{aligned} \sum_{\substack{1\le i\le M+1\\ 1\le j\le N}} \theta_{i,j} y_n(t_i, r_j) = \sum_{\substack{1\le i\le M\\ 1\le k\le K}} \rho^{\omega}_{n,i,k}\, y_n(t_i, v_k) + \sum_{1\le j\le N} \theta_{M+1,j}\, \tilde{y}_n(s_n, r_j) \\ + R_n(a) + \mathcal{O}(n^{-1/4}). \end{aligned} \tag{5.10}$$

Above the spatial points $\{v_k\}$ are a relabeling of $\{r_j, u_\ell\}$, the ω-dependent coefficients $\rho^{\omega}_{n,i,k}$ contain constants $\theta_{i,j}$, probabilities $p^{\omega}_{n,j}(u_\ell n^{1/2}, u_{\ell+1}n^{1/2}\,]$, and zeroes. The constant limits $\rho^{\omega}_{n,i,k} \to \rho_{i,k}$ exist in $\mathbb{P}$-probability as $n \to \infty$. The error in (5.10) is $R_n(a) = \sum_j \theta_{M+1,j} R_{n,j}(a)$.

The variables $\tilde{y}_n(s_n, r_j)$ are functions of the environments $\{\bar{\omega}_m : [nt_{M+1}] \geq m > [nt_M]\}$ and hence independent of $y_n(t_i, v_k)$ for $1 \leq i \leq M$ which are functions of $\{\bar{\omega}_m : [nt_M] \geq m > 0\}$.

On a probability space on which the limit process $\{y(t, r)\}$ has been defined, let $\tilde{y}(t_{M+1} - t_M, \cdot)$ be a random function distributed like $y(t_{M+1} - t_M, \cdot)$ but independent of $\{y(t, r)\}$.

Let f be a bounded Lipschitz continuous function on $\mathbb{R}$, with Lipschitz constant C_f. The goal is to show that the top line (5.11) below vanishes as $n \to \infty$. Add and subtract terms to decompose (5.11) into three differences:

$$\mathbb{E}f\left(\sum_{\substack{1\leq i\leq M+1\\1\leq j\leq N}} \theta_{i,j}\, y_n(t_i, r_j)\right) - Ef\left(\sum_{\substack{1\leq i\leq M+1\\1\leq j\leq N}} \theta_{i,j}\, y(t_i, r_j)\right) \tag{5.11}$$

$$= \left\{ \mathbb{E}f\left(\sum_{\substack{1\leq i\leq M+1\\1\leq j\leq N}} \theta_{i,j}\, y_n(t_i, r_j)\right)\right.$$

$$\left. -\mathbb{E}f\left(\sum_{\substack{1\leq i\leq M\\1\leq k\leq K}} \rho^{\omega}_{n,i,k}\, y_n(t_i, v_k) + \sum_{1\leq j\leq N} \theta_{M+1,j}\, \tilde{y}_n(s_n, r_j)\right)\right\} \tag{5.12}$$

$$+ \left\{ \mathbb{E}f\left(\sum_{\substack{1\leq i\leq M\\1\leq k\leq K}} \rho^{\omega}_{n,i,k}\, y_n(t_i, v_k) + \sum_{1\leq j\leq N} \theta_{M+1,j}\, \tilde{y}_n(s_n, r_j)\right)\right.$$

$$\left. -Ef\left(\sum_{\substack{1\leq i\leq M\\1\leq k\leq K}} \rho_{i,k}\, y(t_i, v_k) + \sum_{1\leq j\leq N} \theta_{M+1,j}\, \tilde{y}(t_{M+1} - t_M, r_j)\right)\right\} \tag{5.13}$$

$$+ \left\{ Ef\left(\sum_{\substack{1\leq i\leq M\\1\leq k\leq K}} \rho_{i,k}\, y(t_i, v_k) + \sum_{1\leq j\leq N} \theta_{M+1,j}\, \tilde{y}(t_{M+1} - t_M, r_j)\right)\right.$$

$$\left. -Ef\left(\sum_{\substack{1\leq i\leq M+1\\1\leq j\leq N}} \theta_{i,j}\, y(t_i, r_j)\right)\right\}. \tag{5.14}$$

The remainder of the proof consists in treating the three differences of expectations (5.12)–(5.14).

By the Lipschitz assumption and (5.10), the difference (5.12) is bounded by

$$C_f \mathbb{E}|R_n(a)| + \mathcal{O}(n^{-1/4}).$$

We need to bound $R_n(a)$. Recall that γ is an $\mathcal{N}(0, \sigma_a^2(t_{M+1} - t_M))$-distribution.

Lemma 5.2. *There exist constants C_1 and a_0 such that, if $a > a_0$, then for any partition $\{u_\ell\}$ of $[-a, a]$ with mesh Δ, and for any $1 \le j \le N$,*

$$\limsup_{n\to\infty} \mathbb{E}|R_{n,j}(a)| \le C_1\big(\sqrt{\Delta} + \gamma(-\infty, -a/2) + \gamma(a/2, \infty)\big).$$

We postpone the proof of Lemma 5.2. From this lemma, given $\varepsilon > 0$, we can choose first a large enough and then Δ small enough so that

$$\limsup_{n\to\infty} [\text{ difference (5.12) }] \le \varepsilon/2.$$

Difference (5.13) vanishes as $n \to \infty$, due to the induction assumption (5.3), the limits $\rho^{\omega}_{n,i,k} \to \rho_{i,k}$ in probability, and the next lemma. Notice that we are not trying to invoke the induction assumption (5.3) for $M + 1$ time points $\{t_1, \ldots, t_M, s_n\}$. Instead, the induction assumption is applied to the first sum inside f in (5.13). To the second sum apply Lemma 5.1, noting that $s_n \to t_{M+1} - t_M$. The two sums are independent of each other, as already observed after (5.10), so they converge jointly. This point is made precise in the next lemma.

Lemma 5.3. *Fix a positive integer k. For each n, let $V_n = (V_n^1, \ldots, V_n^k)$, $X_n = (X_n^1, \ldots, X_n^k)$, and ζ_n be random variables on a common probability space. Assume that X_n and ζ_n are independent of each other for each n. Let v be a constant k-vector, X another random k-vector, and ζ a random variable. Assume the weak limits $V_n \to v$, $X_n \to X$, and $\zeta_n \to \zeta$ hold marginally. Then we have the weak limit*

$$V_n \cdot X_n + \zeta_n \;\to\; v \cdot X + \zeta,$$

where the X and ζ on the right are independent.

To prove this lemma, write

$$V_n \cdot X_n + \zeta_n = (V_n - v) \cdot X_n + v \cdot X_n + \zeta_n$$

and note that since $V_n \to v$ in probability, tightness of $\{X_n\}$ implies that $(V_n - v)\cdot X_n \to 0$ in probability. As mentioned, it applies to show that

$$\lim_{n\to\infty} [\text{ difference (5.13) }] = 0.$$

It remains to examine the difference (5.14). From a consideration of how the coefficients $\rho^{\omega}_{n,i,k}$ in (5.10) arise and from the limit (5.9),

$$\sum_{\substack{1\le i\le M\\1\le k\le K}} \rho_{i,k}\, y(t_i, v_k) + \sum_{1\le j\le N} \theta_{M+1,j}\, \tilde{y}(t_{M+1} - t_M, r_j) = \sum_{\substack{1\le i\le M\\1\le j\le N}} \theta_{i,j} y(t_i, r_j)$$
$$+ \sum_{1\le j\le N} \theta_{M+1,j} \left(\sum_{\ell=0}^{L-1} \gamma(u_\ell - r_j, u_{\ell+1} - r_j] y(t_M, u_\ell) + \tilde{y}(t_{M+1} - t_M, r_j) \right).$$

The first sum after the equality sign matches all but the $(i = M+1)$-terms in the last sum in (5.14). By virtue of the Markov property in (3.8) we can represent the variables $y(t_{M+1}, r_j)$ in the last sum in (5.14) by

$$y(t_{M+1}, r_j) = \int_{\mathbb{R}} \varphi_{\sigma_a^2(t_{M+1}-t_M)}(u - r_j) y(t_M, u)\, du + \tilde{y}(t_{M+1} - t_M, r_j).$$

Then by the Lipschitz property of f it suffices to show that, for each $1 \le j \le N$, the expectation

$$E\left| \int_{\mathbb{R}} \varphi_{\sigma_a^2(t_{M+1}-t_M)}(u - r_j) y(t_M, u)\, du - \sum_{\ell=0}^{L-1} \gamma(u_\ell - r_j, u_{\ell+1} - r_j] y(t_M, u_\ell) \right|$$

can be made small by choice of $a > 0$ and the partition $\{u_\ell\}$. This follows from the moment bounds (3.9) on the increments of the y-process and we omit the details. We have shown that if a is large enough and then Δ small enough,

$$\limsup_{n\to\infty} [\text{ difference (5.14) }] \le \varepsilon/2.$$

To summarize, given bounded Lipschitz f and $\varepsilon > 0$, by choosing $a > 0$ large enough and the partition $\{u_\ell\}$ of $[-a, a]$ fine enough,

$$\limsup_{n\to\infty} \left| \mathbb{E} f\left(\sum_{\substack{1\le i\le M+1\\ 1\le j\le N}} \theta_{i,j} y_n(t_i, r_j) \right) - Ef\left(\sum_{\substack{1\le i\le M+1\\ 1\le j\le N}} \theta_{i,j} y(t_i, r_j) \right) \right| \le \varepsilon.$$

This completes the proof of the induction step and thereby the proof of Theorem 3.2. □

It remains to verify the lemmas that were used along the way.

Proof of Lemma 5.2. We begin with a calculation. Here it is convenient to use the space-time walk $\bar{X}_k^{x,m} = (X_k^{x,m}, m - k)$. First observe that

$$\begin{aligned} E^\omega(X_n^{x,m}) - x - nV &= \sum_{k=0}^{n-1} E^\omega\big[X_{k+1}^{x,m} - X_k^{x,m} - V\big] \\ &= \sum_{k=0}^{n-1} E^\omega\left[E^{T_{\{\bar{X}_k^{x,m}\}}\omega}(X_1^{0,0} - V)\right] \\ &= \sum_{k=0}^{n-1} E^\omega g(T_{\bar{X}_k^{x,m}}\omega). \end{aligned} \tag{5.15}$$

From this, for $x, y \in \mathbb{Z}$,

$$\begin{aligned}
&\mathbb{E}\Big[\big(\{E^\omega(X_n^{x,n}) - x\} - \{E^\omega(X_n^{y,n}) - y\}\big)^2\Big] \\
&= \mathbb{E}\sum_{k=0}^{n-1}\Big(E^\omega\big[g(T_{\bar X_k^{x,n}}\omega) - g(T_{\bar X_k^{y,n}}\omega)\big]\Big)^2 \\
&\quad + 2\sum_{0\le k<\ell<n}\mathbb{E}E^\omega\big[g(T_{\bar X_k^{x,n}}\omega) - g(T_{\bar X_k^{y,n}}\omega)\big]E^\omega\big[g(T_{\bar X_\ell^{x,n}}\omega) - g(T_{\bar X_\ell^{y,n}}\omega)\big] \\
&\qquad \text{(the cross terms for } k<\ell \text{ vanish)} \\
&= \mathbb{E}\sum_{k=0}^{n-1}\left(\sum_{z,w\in\mathbb{Z}^2} P^\omega\{\bar X_k^{x,n} = z\}P^\omega\{\bar X_k^{y,n} = w\}\big[g(T_z\omega) - g(T_w\omega)\big]\right)^2 \\
&= \mathbb{E}\sum_{k=0}^{n-1}\sum_{z,w,u,v\in\mathbb{Z}^2} P^\omega\{\bar X_k^{x,n} = z\}P^\omega\{\bar X_k^{y,n} = w\}P^\omega\{\bar X_k^{x,n} = u\}P^\omega\{\bar X_k^{y,n} = v\} \\
&\quad \times\Big(g(T_z\omega)g(T_u\omega) - g(T_w\omega)g(T_u\omega) - g(T_z\omega)g(T_v\omega) + g(T_w\omega)g(T_v\omega)\Big) \\
&\qquad \big(\text{by independence } \mathbb{E}g(T_z\omega)g(T_u\omega) = \sigma_D^2\mathbf{1}_{\{z=u\}}\big) \\
&= \sigma_D^2\sum_{k=0}^{n-1}\Big(P\{X_k^{x,n} = \widetilde X_k^{x,n}\} - 2P\{X_k^{x,n} = \widetilde X_k^{y,n}\} + P\{X_k^{y,n} = \widetilde X_k^{y,n}\}\Big) \\
&= 2\sigma_D^2\sum_{k=0}^{n-1}\Big(P_0\{Y_k = 0\} - P_{x-y}\{Y_k = 0\}\Big) \\
&= 2\sigma_D^2\big(G_{n-1}(0,0) - G_{n-1}(x-y,0)\big).
\end{aligned}$$

On the last three lines above, as elsewhere in the paper, we used these conventions: X_k and $\widetilde X_k$ denote walks that are independent in a common environment ω, $Y_k = X_k - \widetilde X_k$ is the difference walk, and $G_n(x, y)$ the Green function of Y_k. By Lemma 4.2 we get the inequality

$$\mathbb{E}\Big[\big(\{E^\omega(X_n^{x,n}) - x\} - \{E^\omega(X_n^{y,n}) - y\}\big)^2\Big] \le C\,|x-y| \tag{5.16}$$

valid for all n and all $x, y \in \mathbb{Z}$.

Turning to $R_{n,j}(a)$ defined in (5.6)–(5.7), and utilizing independence,

$$\begin{aligned}
\mathbb{E}|R_{n,j}(a)| \le n^{-1/4}&\sum_{z=\lfloor -a\sqrt{n}\rfloor+1}^{\lfloor a\sqrt{n}\rfloor}\mathbb{E}[p_{n,j}^\omega(z)]\Big(\mathbb{E}\Big[\Big(\Big\{E^\omega\big(X_{\lfloor nt_M\rfloor}^{\lfloor nt_M b\rfloor+z,\lfloor nt_M\rfloor}\big) - z\Big\} \\
&-\ \Big\{E^\omega\big(X_{\lfloor nt_M\rfloor}^{\lfloor nt_M b\rfloor+\lfloor u(z)\sqrt{n}\rfloor,\lfloor nt_M\rfloor}\big) - \lfloor u(z)\sqrt{n}\rfloor\Big\}\Big)^2\Big]\Big)^{1/2} \\
&+n^{-1/4}\sum_{\substack{z\le -a\sqrt{n} \\ z>a\sqrt{n}}}\mathbb{E}[p_{n,j}^\omega(z)]\Big(\mathbb{E}\big[\big(E^\omega(X_{\lfloor nt_M\rfloor}^{\lfloor nt_M b\rfloor+z,\lfloor nt_M\rfloor}) \\
&-E(X_{\lfloor nt_M\rfloor}^{\lfloor nt_M b\rfloor+z,\lfloor nt_M\rfloor})\big)^2\big]\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+n^{-1/4} \sum_{\substack{z \le -a\sqrt{n} \\ z > a\sqrt{n}}} \mathbb{E}[p^{\omega}_{n,j}(z)] \cdot \left| E(X^{\lfloor nt_M b\rfloor + z, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor}) - z \right| \\
&\le C n^{-1/4} \max_{-a\sqrt{n} < z \le a\sqrt{n}} |z - \lfloor u(z)\sqrt{n}\rfloor|^{1/2} \\
&\quad + CP\left\{ \left| X^{\lfloor nt_{M+1} b\rfloor + \lfloor r_j\sqrt{n}\rfloor, \lfloor nt_{M+1}\rfloor}_{\lfloor nt_{M+1}\rfloor - \lfloor nt_M\rfloor} - \lfloor nt_M b\rfloor \right| \ge a\sqrt{n} \right\} + Cn^{-1/4}.
\end{aligned}
$$

For the last inequality above we used (5.16), bound (3.4) on the variance of the quenched mean, and then

$$
E\left(X^{\lfloor nt_M b\rfloor + z, \lfloor nt_M\rfloor}_{\lfloor nt_M\rfloor} \right) - z = \lfloor nt_M b\rfloor + \lfloor nt_M\rfloor V = \lfloor nt_M b\rfloor - \lfloor nt_M\rfloor b = \mathcal{O}(1).
$$

By the choice of $u(z)$, and by the central limit theorem if $a > 2|r_j|$, the limit of the bound on $\mathbb{E}|R_{n,j}(a)|$ as $n \to \infty$ is $C(\sqrt{\Delta} + \gamma(-\infty, -a/2) + \gamma(a/2, \infty))$. This completes the proof of Lemma 5.2. □

Proof of Lemma 5.1. We drop the subscript from t_n and write simply t. For the main part of the proof the only relevant property is that $nt_n = O(n)$. We point this out after the preliminaries.

We show convergence of the linear combination $\sum_{i=1}^{N} \theta_i y_n(t, r_i)$ for an arbitrary but fixed N-vector $\theta = (\theta_1, \ldots, \theta_N)$. This in turn will come from a martingale central limit theorem. For this proof abbreviate $X^i_k = X^{\lfloor ntb\rfloor + \lfloor r_i\sqrt{n}\rfloor, \lfloor nt\rfloor}_k$. For $1 \le k \le \lfloor nt\rfloor$ define

$$
z_{n,k} = n^{-1/4} \sum_{i=1}^{N} \theta_i E^{\omega} g(T_{\bar{X}^i_{k-1}} \omega)
$$

so that by (5.15)

$$
\sum_{k=1}^{\lfloor nt\rfloor} z_{n,k} = \sum_{i=1}^{N} \theta_i y_n(t, r_i) + \mathcal{O}(n^{-1/4}).
$$

The error is deterministic and comes from the discrepancy (3.7) in the centering. It vanishes in the limit and so can be ignored.

A probability of the type $P^{\omega}(X^i_{k-1} = y)$ is a function of the environments

$$
\left\{ \bar{\omega}_j : \lfloor nt\rfloor - k + 2 \le j \le \lfloor nt\rfloor \right\}
$$

while $g(T_{y,s}\omega)$ is a function of $\bar{\omega}_s$. For a fixed n, $\{z_{n,k} : 1 \le k \le \lfloor nt\rfloor\}$ are martingale differences with respect to the filtration

$$
\mathcal{U}_{n,k} = \sigma\left\{ \bar{\omega}_j : \lfloor nt\rfloor - k + 1 \le j \le \lfloor nt\rfloor \right\} \qquad (1 \le k \le \lfloor nt\rfloor)
$$

with $\mathcal{U}_{n,0}$ equal to the trivial σ-algebra. The goal is to show that $\sum_{k=1}^{\lfloor nt\rfloor} z_{n,k}$ converges to a centered Gaussian with variance $\sum_{1\le i,j\le N} \theta_i\theta_j \Gamma_q((t, r_i), (t, r_j))$. By the Lindeberg-Feller Theorem for martingales, it suffices to check that

$$
\sum_{k=1}^{\lfloor nt\rfloor} \mathbb{E}\left[z^2_{n,k} \,\middle|\, \mathcal{U}_{n,k-1} \right] \longrightarrow \sum_{1\le i,j\le N} \theta_i\theta_j \Gamma_q((t, r_i), (t, r_j)) \tag{5.17}
$$

and

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[z_{n,k}^2 \mathbf{1}\{|z_{n,k}| \geq \varepsilon\} \,\big|\, \mathcal{U}_{n,k-1}\big] \longrightarrow 0 \tag{5.18}$$

in probability, as $n \to \infty$, for every $\varepsilon > 0$. Condition (5.18) is trivially satisfied because $|z_{n,k}| \leq Cn^{-1/4}$ by the boundedness of g.

The main part of the proof consists of checking (5.17). This argument is a generalization of the proof of [14, Theorem 4.1] where it was done for a nearest-neighbor walk. We follow their reasoning for the first part of the proof. Since $\sigma_D^2 = \mathbb{E}[g^2]$ and since conditioning $z_{n,k}^2$ on $\mathcal{U}_{n,k-1}$ entails integrating out the environments $\bar{\omega}_{\lfloor nt \rfloor - k+1}$, one can derive

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[z_{n,k}^2 \,|\, \mathcal{U}_{n,k-1}] = \sigma_D^2 \sum_{1 \leq i,j \leq N} \theta_i \theta_j \, n^{-1/2} \sum_{k=0}^{\lfloor nt \rfloor -1} P^{\omega}(X_k^i = \widetilde{X}_k^j),$$

where X_k^i and $\widetilde{X}_k^j$ are two walks independent under the common environment ω, started at $(\lfloor ntb \rfloor + \lfloor r_i \sqrt{n} \rfloor, \lfloor nt \rfloor)$ and $(\lfloor ntb \rfloor + \lfloor r_j \sqrt{n} \rfloor, \lfloor nt \rfloor)$.

By (4.5),

$$\sigma_D^2 n^{-1/2} \sum_{k=0}^{\lfloor nt \rfloor -1} P(X_k^i = \widetilde{X}_k^j) \longrightarrow \Gamma_q((t, r_i), (t, r_j)). \tag{5.19}$$

This limit holds if instead of a fixed t on the left we have a sequence $t_n \to t$. Consequently we will have proved (5.17) if we show, for each fixed pair (i, j), that

$$n^{-1/2} \sum_{k=0}^{\lfloor nt \rfloor -1} \Big(P^{\omega}\{X_k^i = \widetilde{X}_k^j\} - P\{X_k^i = \widetilde{X}_k^j\} \Big) \longrightarrow 0 \tag{5.20}$$

in $\mathbb{P}$-probability. For the above statement the behavior of t is immaterial as long as it stays bounded as $n \to \infty$.

Rewrite the expression in (5.20) as

$$\begin{aligned}
&n^{-1/2} \sum_{k=0}^{\lfloor nt \rfloor -1} \Big(P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,k}\} - P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,0}\} \Big) \\
&= n^{-1/2} \sum_{k=0}^{\lfloor nt \rfloor -1} \sum_{\ell=0}^{k-1} \Big(P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,\ell+1}\} - P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,\ell}\} \Big) \\
&= n^{-1/2} \sum_{\ell=0}^{\lfloor nt \rfloor -1} \sum_{k=\ell+1}^{\lfloor nt \rfloor -1} \Big(P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,\ell+1}\} - P\{X_k^i = \widetilde{X}_k^j \,|\, \mathcal{U}_{n,\ell}\} \Big) \\
&\equiv n^{-1/2} \sum_{\ell=0}^{\lfloor nt \rfloor -1} R_\ell,
\end{aligned}$$

where the last line defines R_ℓ. Check that $\mathbb{E}R_\ell R_m = 0$ for $\ell \neq m$. Thus it is convenient to verify our goal (5.20) by checking L^2 convergence, in other words by showing

$$\begin{aligned} &n^{-1}\sum_{\ell=0}^{\lfloor nt\rfloor-1}\mathbb{E}[R_\ell^2] \\ &= n^{-1}\sum_{\ell=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\left\{\sum_{k=\ell+1}^{\lfloor nt\rfloor-1}\big(P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell+1}\}-P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell}\}\big)\right\}^2\right] \\ &\longrightarrow 0. \end{aligned} \tag{5.21}$$

For the moment we work on a single term inside the braces in (5.21), for a fixed pair $k > \ell$. Write $Y_m = X_m^i - \widetilde{X}_m^j$ for the difference walk. By the Markov property of the walks [recall (2.27)] we can write

$$\begin{aligned} P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell+1}\} = &\sum_{x,\tilde{x},y,\tilde{y}\in\mathbb{Z}} P^\omega\{X_\ell^i=x,\,\widetilde{X}_\ell^j=\tilde{x}\} \\ &\times u^\omega_{\lfloor nt\rfloor-\ell}(x,y-x)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},\tilde{y}-\tilde{x})P(Y_k=0\,|\,Y_{\ell+1}=y-\tilde{y}) \end{aligned}$$

and similarly for the other conditional probability

$$\begin{aligned} P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell}\} = &\sum_{x,\tilde{x},y,\tilde{y}\in\mathbb{Z}} P^\omega\{X_\ell^i=x,\,\widetilde{X}_\ell^j=\tilde{x}\} \\ &\times \mathbb{E}[u^\omega_{\lfloor nt\rfloor-\ell}(x,y-x)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},\tilde{y}-\tilde{x})]P(Y_k=0\,|\,Y_{\ell+1}=y-\tilde{y}). \end{aligned}$$

Introduce the transition probability $q(x,y)$ of the Y-walk. Combine the above decompositions to express the (k,ℓ) term inside the braces in (5.21) as

$$\begin{aligned} &P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell+1}\}-P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell}\} \\ &= \sum_{x,\tilde{x},y,\tilde{y}\in\mathbb{Z}} P^\omega\{X_\ell^i=x,\,\widetilde{X}_\ell^j=\tilde{x}\}q^{k-\ell-1}(y-\tilde{y},0) \\ &\quad\times\big(u^\omega_{\lfloor nt\rfloor-\ell}(x,y-x)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},\tilde{y}-\tilde{x}) \\ &\quad-\mathbb{E}[u^\omega_{\lfloor nt\rfloor-\ell}(x,y-x)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},\tilde{y}-\tilde{x})]\big) \\ &= \sum_{x,\tilde{x}} P^\omega\{X_\ell^i=x,\,\widetilde{X}_\ell^j=\tilde{x}\}\sum_{\substack{z,w:\,-M\le w\le M\\ -M\le w-z\le M}} q^{k-\ell-1}(x-\tilde{x}+z,0) \\ &\quad\times\Big(u^\omega_{\lfloor nt\rfloor-\ell}(x,w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z) \\ &\quad-\mathbb{E}[u^\omega_{\lfloor nt\rfloor-\ell}(x,w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z)]\Big). \end{aligned} \tag{5.22}$$

The last sum above uses the finite range M of the jump probabilities. Introduce the quantities

$$\rho_\ell^\omega(x,x+m) = \sum_{y:y\le m} u^\omega_{\lfloor nt\rfloor-\ell}(x,y) = \sum_{y=-M}^{m} u^\omega_{\lfloor nt\rfloor-\ell}(x,y)$$

and

$$\zeta_\ell^\omega(x,\tilde{x},z,w) = \rho_\ell^\omega(x,x+w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z)$$
$$-\mathbb{E}\big[\rho_\ell^\omega(x,x+w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z)\big].$$

Fix $(x,\tilde{x})$, consider the sum over z and w on line (5.22), and continue with a "summation by parts" step:

$$\sum_{\substack{z,w:\,-M\le w\le M\\ -M\le w-z\le M}} q^{k-\ell-1}(x-\tilde{x}+z,0)\Big(u^\omega_{\lfloor nt\rfloor-\ell}(x,w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z)$$
$$-\mathbb{E}[u^\omega_{\lfloor nt\rfloor-\ell}(x,w)u^\omega_{\lfloor nt\rfloor-\ell}(\tilde{x},w-z)]\Big)$$
$$=\sum_{\substack{z,w:\,-M\le w\le M\\ -M\le w-z\le M}} q^{k-\ell-1}(x-\tilde{x}+z,0)\big(\zeta_\ell^\omega(x,\tilde{x},z,w)-\zeta_\ell^\omega(x,\tilde{x},z-1,w-1)\big)$$
$$=\sum_{\substack{z,w:\,-M\le w\le M\\ -M\le w-z\le M}} \Big(q^{k-\ell-1}(x-\tilde{x}+z,0)-q^{k-\ell-1}(x-\tilde{x}+z+1,0)\Big)\zeta_\ell^\omega(x,\tilde{x},z,w)$$
$$+\sum_{z=0}^{2M} q^{k-\ell-1}(x-\tilde{x}+z+1,0)\zeta_\ell^\omega(x,\tilde{x},z,M)$$
$$-\sum_{z=-2M-1}^{-1} q^{k-\ell-1}(x-\tilde{x}+z+1,0)\zeta_\ell^\omega(x,\tilde{x},z,-M-1).$$

By definition of the range M, the last sum above vanishes because $\zeta_\ell^\omega(x,\tilde{x},z,-M-1)=0$. Take this into consideration, substitute the last form above into (5.22) and sum over $k=\ell+1,\dots,\lfloor nt\rfloor-1$. Define the quantity

$$A_{\ell,n}(x)=\sum_{k=\ell+1}^{\lfloor nt\rfloor-1}\big(q^{k-\ell-1}(x,0)-q^{k-\ell-1}(x+1,0)\big). \tag{5.23}$$

Then the expression in braces in (5.21) is represented as

$$R_\ell=\sum_{k=\ell+1}^{\lfloor nt\rfloor-1}\big(P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell+1}\}-P\{X_k^i=\widetilde{X}_k^j\,|\,\mathcal{U}_{n,\ell}\}\big)$$
$$=\sum_{x,\tilde{x}}P^\omega\{X_\ell^i=x,\widetilde{X}_\ell^j=\tilde{x}\}\sum_{\substack{z,w:\,-M\le w\le M\\ -M\le w-z\le M}} A_{\ell,n}(x-\tilde{x}+z)\zeta_\ell^\omega(x,\tilde{x},z,w) \tag{5.24}$$
$$+\sum_{x,\tilde{x}}P^\omega\{X_\ell^i=x,\widetilde{X}_\ell^j=\tilde{x}\}\sum_{z=0}^{2M}\sum_{k=\ell+1}^{\lfloor nt\rfloor-1}q^{k-\ell-1}(x-\tilde{x}+z+1,0)\zeta_\ell^\omega(x,\tilde{x},z,M) \tag{5.25}$$
$$\equiv R_{\ell,1}+R_{\ell,2},$$

where $R_{\ell,1}$ and $R_{\ell,2}$ denote the sums on lines (5.24) and (5.25).

Recall from (5.21) that our goal was to show that $n^{-1}\sum_{\ell=0}^{\lfloor nt\rfloor-1}\mathbb{E}R_\ell^2 \to 0$ as $n\to\infty$. We show this separately for $R_{\ell,1}$ and $R_{\ell,2}$.

As a function of ω, $\zeta_\ell^\omega(\cdots)$ is a function of $\bar{\omega}_{\lfloor nt\rfloor-\ell}$ and hence independent of the probabilities on line (5.24). Thus we get

$$\begin{aligned}\mathbb{E}[R_{\ell,1}^2] = \sum_{x,\tilde{x},x',\tilde{x}'} &\mathbb{E}\big[P^\omega\{X_\ell^i = x, \widetilde{X}_\ell^j = \tilde{x}\}P^\omega\{X_\ell^i = x', \widetilde{X}_\ell^j = \tilde{x}'\}\big]\\ &\times \sum_{\substack{-M\le w\le M\\ -M\le w-z\le M}} \sum_{\substack{-M\le w'\le M\\ -M\le w'-z'\le M}} A_{\ell,n}(x-\tilde{x}+z)A_{\ell,n}(x'-\tilde{x}'+z')\\ &\times \mathbb{E}\big[\zeta_\ell^\omega(x,\tilde{x},z,w)\zeta_\ell^\omega(x',\tilde{x}',z',w')\big].\end{aligned} \tag{5.26}$$

Lemma 4.2 implies that $A_{\ell,n}(x)$ is uniformly bounded over (ℓ,n,x). Random variable $\zeta_\ell^\omega(x,\tilde{x},z,w)$ is mean zero and a function of the environments $\{\omega_{x,\lfloor nt\rfloor-\ell}, \omega_{\tilde{x},\lfloor nt\rfloor-\ell}\}$. Consequently the last expectation on line (5.26) vanishes unless $\{x,\tilde{x}\}\cap\{x',\tilde{x}'\}\ne\emptyset$. The sums over z, w, z', w' contribute a constant because of their bounded range. Taking all these into consideration, we obtain the bound

$$\mathbb{E}[R_{\ell,1}^2] \le C\Big(P\{X_\ell^i = \widetilde{X}_\ell^i\} + P\{X_\ell^i = \widetilde{X}_\ell^j\} + P\{X_\ell^j = \widetilde{X}_\ell^j\}\Big). \tag{5.27}$$

By (4.5) we get the bound

$$n^{-1}\sum_{\ell=0}^{\lfloor nt\rfloor-1}\mathbb{E}[R_{\ell,1}^2] \le Cn^{-1/2} \tag{5.28}$$

which vanishes as $n\to\infty$.

For the remaining sum $R_{\ell,2}$ observe first that

$$\zeta_\ell^\omega(x,\tilde{x},z,M) = u_{\lfloor nt\rfloor-\ell}^\omega(\tilde{x}, M-z) - \mathbb{E}u_{\lfloor nt\rfloor-\ell}^\omega(\tilde{x}, M-z). \tag{5.29}$$

Summed over $0\le z\le 2M$ this vanishes, so we can start by rewriting as follows:

$$\begin{aligned}R_{\ell,2} &= \sum_{x,\tilde{x}} P^\omega\{X_\ell^i = x, \widetilde{X}_\ell^j = \tilde{x}\}\\ &\quad\times\sum_{z=0}^{2M}\sum_{k=\ell+1}^{\lfloor nt\rfloor-1}\big(q^{k-\ell-1}(x-\tilde{x}+z+1,0) - q^{k-\ell-1}(x-\tilde{x},0)\big)\zeta_\ell^\omega(x,\tilde{x},z,M)\\ &= -\sum_{x,\tilde{x}} P^\omega\{X_\ell^i = x, \widetilde{X}_\ell^j = \tilde{x}\}\sum_{z=0}^{2M}\sum_{m=0}^{z} A_{\ell,n}(x-\tilde{x}+m,0)\zeta_\ell^\omega(x,\tilde{x},z,M)\\ &= -\sum_{x,\tilde{x}} P^\omega\{X_\ell^i = x, \widetilde{X}_\ell^j = \tilde{x}\}\sum_{m=0}^{2M} A_{\ell,n}(x-\tilde{x}+m,0)\bar{\rho}_\ell^\omega(\tilde{x},\tilde{x}+M-m),\end{aligned}$$

where we abbreviated on the last line

$$\bar{\rho}_\ell^\omega(\tilde{x},\tilde{x}+M-m) = \rho_\ell^\omega(\tilde{x},\tilde{x}+M-m) - \mathbb{E}\rho_\ell^\omega(\tilde{x},\tilde{x}+M-m).$$

Square the last representation for $R_{\ell,2}$, take $\mathbb{E}$-expectation, and note that

$$\mathbb{E}\big[\bar\rho_\ell^\omega(\tilde x, \tilde x + M - m)\bar\rho_\ell^\omega(\tilde x', \tilde x' + M - m')\big] = 0$$

unless $\tilde x = \tilde x'$. Thus the reasoning applied to $R_{\ell,1}$ can be repeated, and we conclude that also $n^{-1}\sum_{\ell=0}^{\lfloor nt\rfloor-1}\mathbb{E}R_{\ell,2}^2 \to 0$.

To summarize, we have verified (5.21), thereby (5.20) and condition (5.17) for the martingale CLT. This completes the proof of Lemma 5.1. □

6. Proofs for Forward Walks in a Random Environment

6.1. Proof of Theorem 3.4. The proof of Theorem 3.4 is organized in the same way as the proof of Theorem 3.2 so we restrict ourselves to a few remarks. The Markov property reads now ($0 \le s < t, r \in \mathbb{R}$):

$$\begin{aligned} a_n(t,r) = a_n(s,r) + \sum_{y\in\mathbb{Z}} P^\omega\Big\{Z_{\lfloor ns\rfloor}^{\lfloor r\sqrt{n}\rfloor,0} = \lfloor r\sqrt{n}\rfloor + \lfloor nsV\rfloor + y\Big\} \\ \times n^{-1/4}\Big\{E^\omega\Big(Z_{\lfloor nt\rfloor-\lfloor ns\rfloor}^{\lfloor r\sqrt{n}\rfloor+\lfloor nsV\rfloor+y,\lfloor ns\rfloor}\Big) - \lfloor r\sqrt{n}\rfloor - y - \lfloor nt\rfloor V\Big\} \\ + n^{-1/4}\big(\lfloor ns\rfloor V - \lfloor nsV\rfloor\big). \end{aligned}$$

This serves as the basis for the inductive proof along time levels, exactly as done in the argument following (5.3).

Lemma 5.1 about the convergence at a fixed t-level applies to $a_n(t,\cdot)$ exactly as worded. This follows from noting that, up to a trivial difference from integer parts, the processes $a_n(t,\cdot)$ and $y_n(t,\cdot)$ are the same. Precisely, if S denotes the $\mathbb{P}$-preserving transformation on Ω defined by $(S\omega)_{x,\tau} = \omega_{-\lfloor ntb\rfloor+x,\lfloor nt\rfloor-\tau}$, then

$$E^{S\omega}(X_{\lfloor nt\rfloor}^{\lfloor ntb\rfloor+\lfloor r\sqrt{n}\rfloor,\lfloor nt\rfloor}) - \lfloor r\sqrt{n}\rfloor = E^\omega(Z_{\lfloor nt\rfloor}^{\lfloor r\sqrt{n}\rfloor,0}) - \lfloor r\sqrt{n}\rfloor + \lfloor ntb\rfloor.$$

The errors in the inductive argument are treated with the same arguments as used in Lemma 5.2 to treat $R_{n,j}(a)$.

6.2. Proof of Corollary 3.5. We start with a moment bound that will give tightness of the processes.

Lemma 6.1. *There exists a constant* $0 < C < \infty$ *such that, for all* $n \in \mathbb{N}$,

$$\mathbb{E}\big[(E^\omega(Z_n^{0,0}) - nV)^6\big] \le Cn^{3/2}.$$

Proof. From

$$\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_n^{x,0}}\omega) - E^\omega g(T_{\bar Z_n^{0,0}}\omega)\big)^2\Big] = 2\sigma_D^2\big(P[Y_n^0 = 0] - P[Y_n^x = 0]\big)$$

we get

$$P[Y_n^x = 0] \le P[Y_n^0 = 0] \quad \text{for all } n \ge 0 \text{ and } x \in \mathbb{Z}. \tag{6.1}$$

Abbreviate $\bar Z_n = \bar Z_n^{0,0}$ for this proof. $E^\omega(Z_n) - nV$ is a mean-zero martingale with increments $E^\omega g(T_{\bar Z_k}\omega)$ relative to the filtration $\mathcal{H}_n = \sigma\{\bar\omega_k : 0 \le k < n\}$. By the Burkholder-Davis-Gundy inequality [7],

$$\mathbb{E}\big[(E^\omega(Z_n)-nV)^6\big]\le C\mathbb{E}\bigg[\Big(\sum_{k=0}^{n-1}\big[E^\omega g(T_{\bar Z_k}\omega)\big]^2\Big)^3\bigg].$$

Expanding the cube yields four sums

$$\begin{aligned}&C\sum_{0\le k<n}\mathbb{E}\big[\big(E^\omega g(T_{\bar Z_k}\omega)\big)^6\big]+C\sum_{0\le k_1<k_2<n}\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^4\Big]\\ &+C\sum_{0\le k_1<k_2<n}\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^4\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\Big]\\ &+C\sum_{0\le k_1<k_2<k_3<n}\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_3}}\omega)\big)^2\Big]\end{aligned}$$

with a constant C that bounds the number of arrangements of each type. Replacing some g-factors with constant upper bounds simplifies the quantity to this:

$$\begin{aligned}&C\sum_{0\le k<n}\mathbb{E}\big[\big(E^\omega g(T_{\bar Z_k}\omega)\big)^2\big]+C\sum_{0\le k_1<k_2<n}\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\Big]\\ &+C\sum_{0\le k_1<k_2<k_3<n}\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_3}}\omega)\big)^2\Big].\end{aligned}$$

The expression above is bounded by $C(n^{1/2}+n+n^{3/2})$. We show the argument for the last sum of triple products. (Same reasoning applies to the first two sums.) It utilizes repeatedly independence, $\mathbb{E}g(T_u\omega)g(T_v\omega)=\sigma_D^2\mathbf{1}_{\{u=v\}}$ for $u,v\in\mathbb{Z}^2$, and (6.1). Fix $0\le k_1<k_2<k_3<n$. Let $\bar Z'_k$ denote an independent copy of the walk $\bar Z_k$ in the same environment ω:

$$\begin{aligned}&\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_3}}\omega)\big)^2\Big]\\ &=\mathbb{E}\bigg[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\\ &\qquad\times\sum_{u,v\in\mathbb{Z}^2}P^\omega\{\bar Z_{k_3}=u\}P^\omega\{\bar Z'_{k_3}=v\}\mathbb{E}\big\{g(T_u\omega)g(T_v\omega)\big\}\bigg]\\ &=C\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2P^\omega\{\bar Z_{k_3}=\bar Z'_{k_3}\}\Big]\\ &=C\mathbb{E}\bigg[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\\ &\qquad\times\sum_{u,v\in\mathbb{Z}^2}P^\omega\{\bar Z_{k_2+1}=u,\bar Z'_{k_2+1}=v\}\bigg]\mathbb{E}P^\omega\{\bar Z^u_{k_3-k_2-1}=\bar Z^v_{k_3-k_2-1}\}\\ &\qquad\text{(walks } \bar Z^u_k \text{ and } \bar Z^v_k \text{ are independent under a common } \omega)\\ &\le C\mathbb{E}\bigg[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\\ &\qquad\times\sum_{u,v\in\mathbb{Z}^2}P^\omega\{\bar Z_{k_2+1}=u,\bar Z'_{k_2+1}=v\}\bigg]P(Y^0_{k_3-k_2-1}=0)\\ &=C\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\Big]P(Y^0_{k_3-k_2-1}=0).\end{aligned}$$

Now repeat the same step, and ultimately arrive at

$$\begin{aligned}\sum_{0\le k_1<k_2<k_3<n} &\mathbb{E}\Big[\big(E^\omega g(T_{\bar Z_{k_1}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_2}}\omega)\big)^2\big(E^\omega g(T_{\bar Z_{k_3}}\omega)\big)^2\Big]\\ &\le C\sum_{0\le k_1<k_2<k_3<n} P(Y^0_{k_1}=0)P(Y^0_{k_2-k_1-1}=0)P(Y^0_{k_3-k_2-1}=0)\\ &\le C\cdot G_{n-1}(0,0)^3\le Cn^{3/2}.\end{aligned}$$

□

By Theorem 8.8 in [12, Chap. 3],

$$\mathbb{E}\big[\big(a_n(t+h,r)-a_n(t,r)\big)^3\big(a_n(t,r)-a_n(t-h,r)\big)^3\big]\le Ch^{3/2}$$

is sufficient for tightness of the processes $\{a_n(t,r): t\ge 0\}$. The left-hand side above is bounded by

$$\mathbb{E}\big[\big(a_n(t+h,r)-a_n(t,r)\big)^6\big]+\mathbb{E}\big[\big(a_n(t,r)-a_n(t-h,r)\big)^6\big].$$

Note that if $h<1/(2n)$ then $\big(a_n(t+h,r)-a_n(t,r)\big)\big(a_n(t,r)-a_n(t-h,r)\big)=0$ due to the discrete time of the unscaled walks, while if $h\ge 1/(2n)$ then $\lfloor n(t+h)\rfloor-\lfloor nt\rfloor\le 3nh$. Putting these points together shows that tightness will follow from the next moment bound.

Lemma 6.2. *There exists a constant* $0<C<\infty$ *such that, for all* $0\le m<n\in\mathbb{N}$,

$$\mathbb{E}\big[\big(\{E^\omega(Z^{0,0}_n)-nV\}-\{E^\omega(Z^{0,0}_m)-mV\}\big)^6\big]\le C(n-m)^{3/2}.$$

Proof. The claim reduces to Lemma 6.1 by restarting the walks at time m. □

Convergence of finite-dimensional distributions in Corollary 3.5 follows from Theorem 3.4. The limiting process $\bar a(\cdot)=\lim a_n(\cdot,r)$ is identified by its covariance $E\bar a(s)\bar a(t)=\Gamma_q\big((s\wedge t,r),(s\wedge t,r)\big)$. This completes the proof of Corollary 3.5.

7. Proofs for the Random Average Process

This section requires Theorem 3.2 from the space-time RWRE section.

7.1. Separation of effects. As the form of the limiting process in Theorem 2.1 suggests, we can separate the fluctuations that come from the initial configuration from those created by the dynamics. The quenched means of the RWRE represent the latter. We start with the appropriate decomposition. Abbreviate

$$x_{n,r}=x(n,r)=\lfloor n\bar y\rfloor+\lfloor r\sqrt{n}\,\rfloor.$$

Recall that we are considering $\bar{y} \in \mathbb{R}$ fixed, while $(t, r) \in \mathbb{R}_+ \times \mathbb{R}$ is variable and serves as the index for the process,

$$\begin{aligned}
\sigma^n_{\lfloor nt \rfloor}(x_{n,r} + \lfloor ntb \rfloor) - \sigma^n_0(x_{n,r}) &= E^\omega\Big[\sigma^n_0(X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}) - \sigma^n_0(x_{n,r})\Big] \\
&= E^\omega\Bigg[\mathbf{1}_{\left\{X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor} > x(n,r)\right\}} \sum_{i=x(n,r)+1}^{X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}} \eta^n_0(i) \\
&\quad -\mathbf{1}_{\left\{X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor} < x(n,r)\right\}} \sum_{i=X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}+1}^{x(n,r)} \eta^n_0(i)\Bigg] \\
&= \sum_{i > x(n,r)} P^\omega\left\{i \le X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\right\} \cdot \eta^n_0(i) \\
&\quad - \sum_{i \le x(n,r)} P^\omega\left\{i > X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\right\} \cdot \eta^n_0(i).
\end{aligned}$$

Recalling the means $\varrho(i/n) = \mathbf{E}\eta^n_0(i)$ we write this as

$$\sigma^n_{\lfloor nt \rfloor}(x_{n,r} + \lfloor ntb \rfloor) - \sigma^n_0(x_{n,r}) = Y^n(t, r) + H^n(t, r), \tag{7.1}$$

where

$$\begin{aligned}
Y^n(t, r) = \sum_{i \in \mathbb{Z}} \big(\eta^n_0(i) - \varrho(i/n)\big)\Big(&\mathbf{1}\{i > x_{n,r}\}P^\omega\left\{i \le X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\right\} \\
&- \mathbf{1}\{i \le x_{n,r}\}P^\omega\big\{i > X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\big\}\Big)
\end{aligned}$$

and

$$\begin{aligned}
H^n(t, r) = \sum_{i \in \mathbb{Z}} \varrho(i/n)\Big(&\mathbf{1}\{i > x_{n,r}\}P^\omega\big\{i \le X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\big\} \\
&- \mathbf{1}\{i \le x_{n,r}\}P^\omega\big\{i > X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\big\}\Big).
\end{aligned}$$

The plan of the proof of Theorem 2.1 is summarized in the next lemma. In the pages that follow we then show the finite-dimensional weak convergence $n^{-1/4}H^n \to H$, and the finite-dimensional weak convergence $n^{-1/4}Y^n \to Y$ for a fixed ω. This last statement is actually not proved quite in the strength just stated, but the spirit is correct. The distributional limit $n^{-1/4}Y^n \to Y$ comes from the centered initial increments $\eta^n_0(i) - \varrho(i/n)$, while a homogenization effect takes place for the coefficients $P^\omega\{i \le X^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}_{\lfloor nt \rfloor}\}$ which converge to limiting deterministic Gaussian probabilities. Since the initial height functions σ^n_0 and the random environments ω that drive the dynamics are independent, we also get convergence $n^{-1/4}(Y^n + H^n) \to Y + H$ with independent terms Y and H. This is exactly the statement of Theorem 2.1.

Lemma 7.1. *Let* $(\Omega_0, \mathcal{F}_0, P_0)$ *be a probability space on which are defined independent random variables* η *and* ω *with values in some abstract measurable spaces. The marginal laws are* $\mathbb{P}$ *for* ω *and* $\mathbf{P}$ *for* η*, and* $\mathbf{P}^\omega = \delta_\omega \otimes \mathbf{P}$ *is the conditional probability distribution of* (ω, η)*, given* ω*. Let* $\mathbf{H}^n(\omega)$ *and* $\mathbf{Y}^n(\omega, \eta)$ *be* $\mathbb{R}^N$*-valued measurable functions of* (ω, η)*. Make assumptions* (i)–(ii) *below.*

(i) There exists an $\mathbb{R}^N$*-valued random vector* $\mathbf{H}$ *such that* $\mathbf{H}^n(\omega)$ *converges weakly to* $\mathbf{H}$.
(ii) There exists an $\mathbb{R}^N$*-valued random vector* $\mathbf{Y}$ *such that, for all* $\theta \in \mathbb{R}^N$,

$$\mathbf{E}^\omega[e^{i\theta\cdot\mathbf{Y}^n}] \to E(e^{i\theta\cdot\mathbf{Y}}) \quad \text{in } \mathbb{P}\text{-probability as } n \to \infty.$$

Then $\mathbf{H}^n + \mathbf{Y}^n$ *converges weakly to* $\mathbf{H} + \mathbf{Y}$*, where* $\mathbf{H}$ *and* $\mathbf{Y}$ *are independent.*

Proof. Let θ, λ be arbitrary vectors in $\mathbb{R}^N$. Then

$$\begin{aligned}
&\left|\mathbb{E}\mathbf{E}^\omega\left[e^{i\lambda\cdot\mathbf{H}^n + i\theta\cdot\mathbf{Y}^n}\right] - E\left[e^{i\lambda\cdot\mathbf{H}}\right] E\left[e^{i\theta\cdot\mathbf{Y}}\right]\right| \\
&\quad\le \left|\mathbb{E}\left[e^{i\lambda\cdot\mathbf{H}^n}\left(\mathbf{E}^\omega e^{i\theta\cdot\mathbf{Y}^n} - Ee^{i\theta\cdot\mathbf{Y}}\right)\right]\right| + \left|\left(\mathbb{E}e^{i\lambda\cdot\mathbf{H}^n} - Ee^{i\lambda\cdot\mathbf{H}}\right) Ee^{i\theta\cdot\mathbf{Y}}\right| \\
&\quad\le \left|\mathbb{E}\left[e^{i\lambda\cdot\mathbf{H}^n}\left(\mathbf{E}^\omega e^{i\theta\cdot\mathbf{Y}^n} - Ee^{i\theta\cdot\mathbf{Y}}\right)\right]\right| + \left|\mathbb{E}e^{i\lambda\cdot\mathbf{H}^n} - Ee^{i\lambda\cdot\mathbf{H}}\right|.
\end{aligned}$$

By assumption (i), the second term above goes to 0. By assumption (ii), the integrand in the first term goes to 0 in $\mathbb{P}$-probability. Therefore by bounded convergence the first term goes to 0 as $n \to \infty$. □

Turning to the work itself, we check first that $H^n(t,r)$ can be replaced with a quenched RWRE mean. Then the convergence $H^n \to H$ follows from the RWRE results.

Lemma 7.2. *For any* $S, T < \infty$ *and for* $\mathbb{P}$*-almost every* ω,

$$\lim_{n\to\infty} \sup_{\substack{0\le t\le T\\ -S\le r\le S}} n^{-1/4}\left|H^n(t,r) - \varrho(\bar{y}) \cdot E^\omega\left(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r}\right)\right| = 0.$$

Proof. Decompose $H^n(t,r) = H_1^n(t,r) - H_2^n(t,r)$, where

$$\begin{aligned}
H_1^n(t,r) &= \sum_{i > x(n,r)} P^\omega\left\{i \le X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\} \cdot \varrho(i/n),\\
H_2^n(t,r) &= \sum_{i \le x(n,r)} P^\omega\left\{i > X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\} \cdot \varrho(i/n).
\end{aligned}$$

Working with $H_1^n(t,r)$, we separate out the negligible error,

$$\begin{aligned}
H_1^n(t,r) &= \varrho(\bar{y}) \sum_{i > x(n,r)} P^\omega\left\{i \le X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\}\\
&\qquad + \sum_{i > x(n,r)} P^\omega\left\{i \le X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\} \cdot \left[\varrho(i/n) - \varrho(\bar{y})\right]\\
&= \varrho(\bar{y}) \cdot E^\omega\left[\left(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r}\right)^+\right] + R_1(t,r)
\end{aligned}$$

with

$$R_1(t,r) = \sum_{m=1}^{\infty} P^\omega\left\{x_{n,r} + m \le X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\} \cdot \left[\varrho\left(\frac{x_{n,r}}{n} + \frac{m}{n}\right) - \varrho(\bar{y})\right].$$

Fix a small positive number $\delta < \frac{1}{2}$, and use the boundedness of probabilities and the function ϱ:

$$|R_1(t,r)| \le \sum_{m=1}^{\lfloor n^{1/2+\delta}\rfloor} \left|\varrho\left(\frac{x_{n,r}}{n}+\frac{m}{n}\right)-\varrho(\bar{y})\right| + C\cdot \sum_{m=\lfloor n^{1/2+\delta}\rfloor+1}^{\infty} P^{\omega}\left\{x_{n,r}+m \le X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\right\}. \tag{7.2}$$

By the local Hölder-continuity of ϱ with exponent $\gamma > \frac{1}{2}$, the first sum is $\mathfrak{o}(n^{1/4})$ if $\delta > 0$ is small enough. Since $X_0^{x(n,r)+\lfloor ntb\rfloor,\lfloor nt\rfloor} = x_{n,r} + \lfloor ntb\rfloor$ and by time $\lfloor nt\rfloor$ the walk has displaced by at most $M\lfloor nt\rfloor$, there are at most $\mathcal{O}(n)$ nonzero terms in the second sum in (7.2). Consequently this sum is at most

$$Cn\cdot P^{\omega}\left\{X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r} \ge \lfloor n^{1/2+\delta}\rfloor\right\}.$$

By Lemma 4.3 the last line vanishes uniformly over $t \in [0,T]$ and $r \in [-S,S]$ as $n\to\infty$, for $\mathbb{P}$-almost every ω. We have shown

$$\lim_{n\to\infty} \sup_{\substack{0\le t\le T\\ -S\le r\le S}} n^{-1/4}\left|H_1^n(t,r) - \varrho(\bar{y})\cdot E^{\omega}\left[\left(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r}\right)^{+}\right]\right| = 0 \quad \mathbb{P}\text{-a.s.}$$

Similarly one shows

$$\lim_{n\to\infty} \sup_{\substack{0\le t\le T\\ -S\le r\le S}} n^{-1/4}\left|H_2^n(t,r) - \varrho(\bar{y})\cdot E^{\omega}\left[\left(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r}\right)^{-}\right]\right| = 0 \quad \mathbb{P}\text{-a.s.}$$

The conclusion follows from the combination of these two. □

For a fixed n and $\bar{y}$, the process $E^{\omega}\left(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor} - x_{n,r}\right)$ has the same distribution as the process $y_n(t,r)$ defined in (3.6). A combination of Lemma 7.2 and Theorem 3.2 imply that the finite-dimensional distributions of the processes $n^{-1/4}H_n$ converge weakly, as $n\to\infty$, to the finite-dimensional distributions of the mean-zero Gaussian process H with covariance

$$EH(s,q)H(t,r) = \varrho(\bar{y})^2\Gamma_q((s,q),(t,r)). \tag{7.3}$$

7.2. Finite-dimensional convergence of Y^n. Next we turn to convergence of the finite-dimensional distributions of process Y^n in (7.1). Recall that $B(t)$ is standard Brownian motion, and $\sigma_a^2 = E[(X_1^{0,0}-V)^2]$ is the variance of the annealed walk. Recall the definition

$$\begin{aligned}\Gamma_0((s,q),(t,r)) = &\int_{q\vee r}^{\infty} P[\sigma_a B(s) > x-q]P[\sigma_a B(t) > x-r]\,dx\\ &-\Big\{\mathbf{1}_{\{r>q\}}\int_q^r P[\sigma_a B(s) > x-q]P[\sigma_a B(t) \le x-r]\,dx\\ &+\mathbf{1}_{\{q>r\}}\int_r^q P[\sigma_a B(s) \le x-q]P[\sigma_a B(t) > x-r]\,dx\Big\}\\ &+\int_{-\infty}^{q\wedge r} P[\sigma_a B(s) \le x-q]P[\sigma_a B(t) \le x-r]\,dx.\end{aligned}$$

Recall from (2.9) that $v(\bar{y})$ is the variance of the increments around $\lfloor n\bar{y} \rfloor$. Let $\{Y(t,r) : t \geq 0,\ r \in \mathbb{R}\}$ be a real-valued mean-zero Gaussian process with covariance

$$EY(s,q)Y(t,r) = v(\bar{y})\Gamma_0((s,q),(t,r)). \tag{7.4}$$

Fix N and space-time points $(t_1,r_1),\dots,(t_N,r_N) \in \mathbb{R}_+ \times \mathbb{R}$. Define vectors

$$\mathbf{Y}^n = n^{-1/4}\big(Y^n(t_1,r_1),\dots,Y^n(t_N,r_N)\big) \quad \text{and} \quad \mathbf{Y} = \big(Y(t_1,r_1),\dots,Y(t_N,r_N)\big).$$

This section is devoted to the proof of the next proposition, after which we finish the proof of Theorem 2.1.

Proposition 7.1. *For any vector* $\theta = (\theta_1,\dots,\theta_N) \in \mathbb{R}^N$, $\mathbf{E}^\omega(e^{i\theta\cdot\mathbf{Y}^n}) \to E(e^{i\theta\cdot\mathbf{Y}})$ *in* $\mathbb{P}$*-probability as* $n \to \infty$.

Proof. Let G be a centered Gaussian variable with variance

$$S = v(\bar{y}) \sum_{k,l=1}^{N} \theta_k\theta_l\Gamma_0((t_k,r_k),(t_l,r_l))$$

and so $\theta\cdot\mathbf{Y}$ is distributed like G. We will show that

$$\mathbf{E}^\omega(e^{i\theta\cdot\mathbf{Y}^n}) \to E(e^{iG}) \quad \text{in } \mathbb{P}\text{-probability}. \tag{7.5}$$

Recalling the definition of $Y^n(t,r)$, introduce some notation:

$$\begin{aligned}\zeta_n^\omega(i,t,r) &= \mathbf{1}\{i > x_{n,r}\}P^\omega\big\{i \leq X_{\lfloor nt \rfloor}^{x(n,r)+\lfloor ntb \rfloor,\,\lfloor nt \rfloor}\big\}\\ &\qquad - \mathbf{1}\{i \leq x_{n,r}\}P^\omega\big\{i > X_{\lfloor nt \rfloor}^{x(n,r)+\lfloor ntb \rfloor,\,\lfloor nt \rfloor}\big\}\end{aligned}$$

so that

$$Y^n(t,r) = \sum_{i\in\mathbb{Z}} \big(\eta_0^n(i) - \varrho(i/n)\big)\zeta_n^\omega(i,t,r).$$

Then put

$$\nu_n^\omega(i) = \sum_{k=1}^{N} \theta_k\, \zeta_n^\omega(i,t_k,r_k)$$

and

$$U_n(i) = n^{-1/4}\,\big(\eta_0^n(i) - \varrho(i/n)\big)\,\nu_n^\omega(i).$$

Consequently

$$\theta\cdot\mathbf{Y}^n = \sum_{i\in\mathbb{Z}} U_n(i).$$

To separate out the relevant terms let $\delta > 0$ be small and define

$$W_n = \sum_{i=\lfloor n\bar{y} \rfloor - \lfloor n^{1/2+\delta} \rfloor}^{\lfloor n\bar{y} \rfloor + \lfloor n^{1/2+\delta} \rfloor} U_n(i).$$

For fixed ω and n, under the measure $\mathbf{P}^\omega$ the variables $\{U_n(i)\}$ are constant multiples of centered increments $\eta_0^n(i) - \varrho(i/n)$ and hence independent and mean zero. Recall also that second moments of centered increments $\eta_0^n(i) - \varrho(i/n)$ are uniformly bounded. Thus the terms left out of W_n satisfy

$$\mathbf{E}^\omega\big[(W_n - \theta \cdot \mathbf{Y}^n)^2\big] \le Cn^{-1/2} \sum_{i:|i-\lfloor n\bar{y}\rfloor| > n^{1/2+\delta}} \nu_n^\omega(i)^2,$$

and we wish to show that this upper bound vanishes for $\mathbb{P}$-almost every ω as $n \to \infty$. Using the definition of $\nu_n^\omega(i)$, bounding the sum on the right reduces to bounding sums of the two types

$$n^{-1/2} \sum_{i:|i-\lfloor n\bar{y}\rfloor| > n^{1/2+\delta}} \mathbf{1}\{i > x(n,r_k)\}\Big(P^\omega\big\{i \le X_{\lfloor nt_k\rfloor}^{x(n,r_k)+\lfloor nt_k b\rfloor,\, \lfloor nt_k\rfloor}\big\}\Big)^2$$

and

$$n^{-1/2} \sum_{i:|i-\lfloor n\bar{y}\rfloor| > n^{1/2+\delta}} \mathbf{1}\{i \le x(n,r_k)\}\Big(P^\omega\big\{i > X_{\lfloor nt_k\rfloor}^{x(n,r_k)+\lfloor nt_k b\rfloor,\, \lfloor nt_k\rfloor}\big\}\Big)^2.$$

For large enough n the points $x(n,r_k)$ lie within $\frac{1}{2}n^{1/2+\delta}$ of $\lfloor n\bar{y}\rfloor$, and then the previous sums are bounded by the sums

$$n^{-1/2} \sum_{i \ge x(n,r_k)+(1/2)n^{1/2+\delta}} \Big(P^\omega\big\{i \le X_{\lfloor nt_k\rfloor}^{x(n,r_k)+\lfloor nt_k b\rfloor,\, \lfloor nt_k\rfloor}\big\}\Big)^2$$

and

$$n^{-1/2} \sum_{i \le x(n,r_k)-(1/2)n^{1/2+\delta}} \Big(P^\omega\big\{i > X_{\lfloor nt_k\rfloor}^{x(n,r_k)+\lfloor nt_k b\rfloor,\, \lfloor nt_k\rfloor}\big\}\Big)^2.$$

These vanish for $\mathbb{P}$-almost every ω as $n \to \infty$ by Lemma 4.3, in a manner similar to the second sum in (7.2). Thus $\mathbf{E}^\omega\big[(W_n - \theta \cdot \mathbf{Y}^n)^2\big] \to 0$ and our goal (7.5) has simplified to

$$\mathbf{E}^\omega(e^{iW_n}) \to E(e^{iG}) \quad \text{in } \mathbb{P}\text{-probability.} \tag{7.6}$$

We use the Lindeberg-Feller theorem to formulate conditions for a central limit theorem for W_n under a fixed ω. For Lindeberg-Feller we need to check two conditions:

(LF-i) $$S^n(\omega) \equiv \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} \mathbf{E}^\omega\Big[U_n(i)^2\Big] \underset{n\to\infty}{\longrightarrow} S,$$

(LF-ii) $$\sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} \mathbf{E}^\omega\Big[U_n(i)^2 \cdot \mathbf{1}_{\{|U_n(i)|>\varepsilon\}}\Big] \underset{n\to\infty}{\longrightarrow} 0 \ \text{ for all } \varepsilon > 0.$$

To see that (LF-ii) holds, pick conjugate exponents $p, q > 1$ $(1/p + 1/q = 1)$:

$$\begin{aligned}\mathbf{E}^\omega\left[U_n(i)^2 \cdot \mathbf{1}_{\{U_n(i)^2>\varepsilon^2\}}\right] &\le \left(\mathbf{E}^\omega\left[|U_n(i)|^{2p}\right]\right)^{\frac{1}{p}}\left(\mathbf{P}^\omega\left[U_n(i)^2 > \varepsilon^2\right]\right)^{\frac{1}{q}} \\ &\le \varepsilon^{-\frac{2}{q}}\left(\mathbf{E}^\omega\left[|U_n(i)|^{2p}\right]\right)^{\frac{1}{p}}\left(\mathbf{E}^\omega\left[U_n(i)^2\right]\right)^{\frac{1}{q}} \\ &\le Cn^{-1/2-1/(2q)}.\end{aligned}$$

In the last step we used the bound $|U_n(i)| \le Cn^{-1/4}\left|\eta_0^n(i) - \varrho(i/n)\right|$, boundedness of ϱ, and we took p close enough to 1 to apply assumption (2.11). Condition (LF-ii) follows if $\delta < 1/(2q)$.

We turn to condition (LF-i):

$$\begin{aligned}S^n(\omega) &= \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} \mathbf{E}^\omega\left[U_n(i)^2\right] = \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} n^{-1/2}v(i/n)[\nu_n^\omega(i)]^2 \\ &= \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} n^{-1/2}\left[v(i/n) - v(\bar{y})\right]\left[\nu_n^\omega(i)\right]^2 \\ &\quad + \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} n^{-1/2}v(\bar{y})[\nu_n^\omega(i)]^2.\end{aligned}$$

Due to the local Hölder-property (2.10) of v, the first sum on the last line is bounded above by

$$C(\bar{y})n^{1/2+\delta}n^{-1/2}\left[n^{-1/2+\delta}\right]^\gamma = C(\bar{y})n^{\delta(1+\gamma)-\gamma/2} \to 0$$

for sufficiently small δ. Denote the remaining relevant part by $\tilde{S}^n(\omega)$, given by

$$\begin{aligned}\tilde{S}^n(\omega) &= \sum_{i=\lfloor n\bar{y}\rfloor-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n\bar{y}\rfloor+\lfloor n^{1/2+\delta}\rfloor} n^{-1/2}v(\bar{y})[\nu_n^\omega(i)]^2 = v(\bar{y})n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\left(\nu_n^\omega\left(m + \lfloor n\bar{y}\rfloor\right)\right)^2 \\ &= v(\bar{y})\sum_{k,l=1}^{N}\theta_k\theta_l\, n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\zeta_n^\omega(\lfloor n\bar{y}\rfloor + m, t_k, r_k)\zeta_n^\omega(\lfloor n\bar{y}\rfloor + m, t_l, r_l).\end{aligned} \tag{7.7}$$

Consider for the moment a particular (k, l) term in the first sum on line (7.7). Rename $(s, q) = (t_k, r_k)$ and $(t, r) = (t_l, r_l)$. Expanding the product of the ζ_n^ω-factors gives three sums:

$$n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\zeta_n^{\omega}(\lfloor n\bar{y}\rfloor+m,s,q)\zeta_n^{\omega}(\lfloor n\bar{y}\rfloor+m,t,r)$$

$$= n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\mathbf{1}_{\{m>\lfloor q\sqrt{n}\rfloor\}}\mathbf{1}_{\{m>\lfloor r\sqrt{n}\rfloor\}}P^{\omega}\big(X_{\lfloor ns\rfloor}^{x(n,q)+\lfloor nsb\rfloor,\,\lfloor ns\rfloor}\geq\lfloor n\bar{y}\rfloor+m\big)$$

$$\times P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\geq\lfloor n\bar{y}\rfloor+m\big) \tag{7.8}$$

$$- n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\Big\{\mathbf{1}_{\{m>\lfloor q\sqrt{n}\rfloor\}}\mathbf{1}_{\{m\leq\lfloor r\sqrt{n}\rfloor\}}P^{\omega}\big(X_{\lfloor ns\rfloor}^{x(n,q)+\lfloor nsb\rfloor,\,\lfloor ns\rfloor}\geq\lfloor n\bar{y}\rfloor+m\big)$$

$$\times P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}<\lfloor n\bar{y}\rfloor+m\big)$$

$$+\mathbf{1}_{\{m\leq\lfloor q\sqrt{n}\rfloor\}}\mathbf{1}_{\{m>\lfloor r\sqrt{n}\rfloor\}}P^{\omega}\big(X_{\lfloor ns\rfloor}^{x(n,q)+\lfloor nsb\rfloor,\,\lfloor ns\rfloor}<\lfloor n\bar{y}\rfloor+m\big)$$

$$\times P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\geq\lfloor n\bar{y}\rfloor+m\big)\Big\} \tag{7.9}$$

$$+ n^{-1/2}\sum_{m=-\lfloor n^{1/2+\delta}\rfloor}^{\lfloor n^{1/2+\delta}\rfloor}\mathbf{1}_{\{m\leq\lfloor q\sqrt{n}\rfloor\}}\mathbf{1}_{\{m\leq\lfloor r\sqrt{n}\rfloor\}}P^{\omega}\big(X_{\lfloor ns\rfloor}^{x(n,q)+\lfloor nsb\rfloor,\,\lfloor ns\rfloor}<\lfloor n\bar{y}\rfloor+m\big)$$

$$\times P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}<\lfloor n\bar{y}\rfloor+m\big). \tag{7.10}$$

Each of these three sums (7.8)–(7.10) converges to a corresponding integral in $\mathbb{P}$-probability, due to the quenched CLT Theorem 3.1. To see the correct limit, just note that

$$P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}<\lfloor n\bar{y}\rfloor+m\big)$$
$$= P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}-X_0^{x(n,r)+\lfloor ntb\rfloor,\lfloor nt\rfloor}<-\lfloor ntb\rfloor+m-\lfloor r\sqrt{n}\rfloor\big),$$

and recall that $-b = V$ is the average speed of the walks. We give technical details of the argument for the first sum in the next lemma.

Lemma 7.3. *As $n\to\infty$, the sum in* (7.8) *converges in $\mathbb{P}$-probability to*

$$\int_{q\vee r}^{\infty}P[\sigma_a B(s)>x-q]P[\sigma_a B(t)>x-r]\,dx.$$

Proof of Lemma 7.3. With

$$f_n^{\omega}(x)=P^{\omega}\big(X_{\lfloor ns\rfloor}^{x(n,q)+\lfloor nsb\rfloor,\,\lfloor ns\rfloor}\geq\lfloor n\bar{y}\rfloor+\lfloor x\sqrt{n}\rfloor\big)$$
$$\times P^{\omega}\big(X_{\lfloor nt\rfloor}^{x(n,r)+\lfloor ntb\rfloor,\,\lfloor nt\rfloor}\geq\lfloor n\bar{y}\rfloor+\lfloor x\sqrt{n}\rfloor\big)$$

and

$$I_n^{\omega}=\int_{q\vee r}^{n^{\delta}}f_n^{\omega}(x)dx,$$

the sum in (7.8) equals $I_n^\omega + \mathcal{O}(n^{-1/2})$. By the quenched invariance principle Theorem 3.1, for any fixed x, $f_n^\omega(x)$ converges in $\mathbb{P}$-probability to

$$f(x) = P[\sigma_a B(s) \ge x - q]P[\sigma_a B(t) \ge x - r].$$

We cannot claim this convergence $\mathbb{P}$-almost surely because the walks $X_{\lfloor nt \rfloor}^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor}$ change as n changes. But by a textbook characterization of convergence in probability, for a fixed x each subsequence $n(j)$ has a further subsequence $n(j_\ell)$ such that

$$\mathbb{P}\Big[\omega \,:\, f_{n(j_\ell)}^\omega(x) \underset{\ell\to\infty}{\longrightarrow} f(x)\Big] = 1.$$

By the diagonal trick, one can find one subsequence for all $x \in \mathbb{Q}$ and thus

$$\forall\{n(j)\}, \exists\{j_\ell\} : \mathbb{P}\Big[\omega \,:\, \forall x \in \mathbb{Q} : f_{n(j_\ell)}^\omega(x) \to f(x)\Big] = 1.$$

Since f_n^ω and f are nonnegative and nonincreasing, and f is continuous and decreases to 0, the convergence works for all x and is uniform on $[q \vee r, \infty)$. That is,

$$\forall\{n(j)\}, \exists\{j_\ell\} : \mathbb{P}\Big[\omega \,:\, \Big\| f_{n(j_\ell)}^\omega - f \Big\|_{L^\infty[q\vee r,\,\infty)} \to 0\Big] = 1.$$

It remains to make the step to the convergence of the integral I_n^ω to $\int_{q\vee r}^\infty f(x)\,dx$.

Define now

$$J_n^\omega(A) = \int_{q\vee r}^A f_n^\omega(x)dx.$$

Then, for any $A < \infty$,

$$\forall\{n(j)\}, \exists\{j_\ell\} : \mathbb{P}\Big[\omega \,:\, J_{n(j_\ell)}^\omega(A) \to \int_{q\vee r}^A f(x)dx\Big] = 1.$$

In other words, $J_n^\omega(A)$ converges to $\int_{q\vee r}^A f(x)dx$ in $\mathbb{P}$-probability. Thus, for each $0 < A < \infty$, there is an integer $m(A)$ such that for all $n \ge m(A)$,

$$\mathbb{P}\Big[\omega \,:\, \Big| J_n^\omega(A) - \int_{q\vee r}^A f(x)dx \Big| > A^{-1}\Big] < A^{-1}.$$

Pick $A_n \nearrow \infty$ such that $m(A_n) \le n$. Under the annealed measure P, $X_n^{0,0}$ is a homogeneous mean zero random walk with variance $\mathcal{O}(n)$. Consequently

$$
\begin{aligned}
\mathbb{E}[\,|I_n^\omega - J_n^\omega(A_n)|\,] &\le \int_{A_n \wedge n^\delta}^{\infty} \mathbb{E}[f_n^\omega(x)]dx \\
&\le \int_{A_n \wedge n^\delta}^{\infty} P\Big(X_{\lfloor nt \rfloor}^{x(n,r)+\lfloor ntb \rfloor,\, \lfloor nt \rfloor} \ge x(n,r) - \lfloor r\sqrt{n} \rfloor \\
&\qquad + \lfloor x\sqrt{n} \rfloor\Big)\, dx \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Combine this with

$$
\mathbb{P}\Big[\omega \,:\, \Big|J_n^\omega(A_n) - \int_{q\vee r}^{A_n} f(x)dx\Big| > A_n^{-1}\Big] < A_n^{-1}.
$$

Since $\int_{q\vee r}^{A_n} f(x)dx$ converges to $\int_{q\vee r}^{\infty} f(x)dx$, we have shown that I_n^ω converges to this same integral in $\mathbb{P}$-probability. This completes the proof of Lemma 7.3. □

We return to the main development, the proof of Proposition 7.1. Apply the argument of the lemma to the three sums (7.8)–(7.10) to conclude the following limit in $\mathbb{P}$-probability:

$$
\begin{aligned}
&\lim_{n\to\infty} n^{-1/2} \sum_{m=-\lfloor n^{1/2+\delta} \rfloor}^{\lfloor n^{1/2+\delta} \rfloor} \zeta_n^\omega(\lfloor n\bar{y} \rfloor + m, s, q)\zeta_n^\omega(\lfloor n\bar{y} \rfloor + m, t, r) \\
&= \int_{q\vee r}^{\infty} P[\sigma_a B(s) > x - q]P[\sigma_a B(t) > x - r]\, dx \\
&\quad -\Big\{\mathbf{1}_{\{r>q\}} \int_q^r P[\sigma_a B(s) > x - q]P[\sigma_a B(t) \le x - r]\, dx \\
&\quad +\mathbf{1}_{\{q>r\}} \int_r^q P[\sigma_a B(s) \le x - q]P[\sigma_a B(t) > x - r]\, dx\Big\} \\
&\quad + \int_{-\infty}^{q\wedge r} P[\sigma_a B(s) \le x - q]P[\sigma_a B(t) \le x - r]\, dx \\
&= \Gamma_0((s,q),(t,r)).
\end{aligned}
$$

Return to condition (LF-i) of the Lindeberg-Feller theorem and the definition (7.7) of $\tilde{S}^n(\omega)$. Since $S^n(\omega) - \tilde{S}^n(\omega) \to 0$ as pointed out above (7.7), we have shown that $S^n \to S$ in $\mathbb{P}$-probability. Consequently

$$
\forall\{n(j)\}, \exists\{j_\ell\} : \mathbb{P}\Big[\omega \,:\, S^{n(j_\ell)}(\omega) \to S\Big] = 1.
$$

This can be rephrased as: given any subsequence $\{n(j)\}$, there exists a further subsequence $\{n(j_\ell)\}$ along which conditions (LF-i) and (LF-ii) of the Lindeberg-Feller theorem are satisfied for the array

$$
\Big\{U_{n(j_\ell)}(i) : \lfloor n(j_\ell)\bar{y} \rfloor - \lfloor n(j_\ell)^{1/2+\delta} \rfloor \le i \le \lfloor n(j_\ell)\bar{y} \rfloor + \lfloor n(j_\ell)^{1/2+\delta} \rfloor,\ \ell \ge 1\Big\}
$$

under the measure $\mathbf{P}^\omega$ for $\mathbb{P}$-a.e. ω. This implies that

$$
\forall\{n(j)\}, \exists\{j_\ell\} : \mathbb{P}\Big[\omega \,:\, \mathbf{E}^\omega(e^{iW_{n(j_\ell)}}) \to E(e^{iG})\Big] = 1.
$$

But the last statement characterizes convergence $\mathbf{E}^\omega(e^{iW_n}) \to E(e^{iG})$ in $\mathbb{P}$-probability. As we already showed above that $W_n - \theta \cdot \mathbf{Y}^n \to 0$ in $\mathbf{P}^\omega$-probability $\mathbb{P}$-almost surely, this completes the proof of Proposition 7.1. □

7.3. Proofs of Theorem 2.1 and Proposition 2.2.

Proof of Theorem 2.1. The decomposition (7.1) gives $z_n = n^{-1/4}(Y^n + H^n)$. The paragraph that follows Lemma 7.2 and Proposition 7.1 verify the hypotheses of Lemma 7.1 for H^n and Y^n. Thus we have the limit $z_n \to z \equiv Y + H$ in the sense of convergence of finite-dimensional distributions. Since Y and H are mutually independent mean-zero Gaussian processes, their covariances in (7.3) and (7.4) can be added to give (2.18). □

Proof of Proposition 2.2. The value (2.23) for β can be computed from (2.8), or from the probabilistic characterization (4.4) of β via Example 4.1. If we let u denote a random variable distributed like $u_0(0, -1)$, then we get

$$\beta = \frac{\mathbb{E}u - \mathbb{E}(u^2)}{\mathbb{E}u - (\mathbb{E}u)^2} \quad \text{and} \quad \kappa = \frac{\mathbb{E}(u^2) - (\mathbb{E}u)^2}{\mathbb{E}u - \mathbb{E}(u^2)}.$$

With obvious notational simplifications, the evolution step (2.22) can be rewritten as

$$\eta'(k) - \rho = (1 - u_k)(\eta(k) - \rho) + u_{k-1}(\eta(k-1) - \rho) + (u_{k-1} - u_k)\rho.$$

Square both sides, take expectations, use the independence of all variables $\{\eta(k-1), \eta(k), u_k, u_{k-1}\}$ on the right, and use the requirement that $\eta'(k)$ have the same variance v as $\eta(k)$ and $\eta(k-1)$. The result is the identity

$$v = v(1 - 2\mathbb{E}u + 2\mathbb{E}(u^2)) + 2\rho^2(\mathbb{E}(u^2) - (\mathbb{E}u)^2)$$

from which follows $v = \kappa\rho^2$. The rest of part (b) is a straightforward specialization of (2.18). □

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