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RATIOS OF PARTITION FUNCTIONS FOR THE LOG-GAMMA POLYMER

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We introduce a random walk in random environment associated to an underlying directed polymer model in $1 + 1$ dimensions. This walk is the positive temperature counterpart of the competition interface of percolation and arises as the limit of quenched polymer measures. We prove this limit for the exactly solvable log-gamma polymer, as a consequence of almost sure limits of ratios of partition functions. These limits of ratios give the Busemann functions of the log-gamma polymer, and furnish centered cocycles that solve a variational formula for the limiting free energy. Limits of ratios of point-to-point and point-to-line partition functions manifest a duality between tilt and velocity that comes from quenched large deviations under polymer measures. In the log-gamma case, we identify a family of ergodic invariant distributions for the random walk in random environment.

1. Introduction. In directed polymer models the definition of weak disorder is that normalized point-to-line partition functions converge to a strictly positive random variable. In strong disorder these normalized partition functions converge to zero. Weak disorder takes place only in dimensions $3 + 1$ and higher and under high enough temperature, while lower dimensions are in strong disorder throughout the temperature range; see [3, 5–8, 12, 20] for reviews and some key results.

We work in $1 + 1$ dimensions with the explicitly solvable log-gamma polymer. We show that ratios of both point-to-point partition functions and tilted point-to-line partition functions converge almost surely to gamma-distributed limits. Out of this basic fact we derive several consequences.

(i) Limits of ratios of partition functions give us limits of quenched polymer measures, both point-to-point and point-to-line, as the path length tends to infinity.

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The limit processes can be regarded as infinitely long polymers. Technically they are random walks in correlated random environments (RWRE). When we average over the environment, this RWRE has fluctuation exponent $2/3$, in accordance with $1+1$ dimensional Kardar–Parisi–Zhang universality. This polymer RWRE is also a positive temperature counterpart of a competition interface in a percolation model. (This terminology comes from the idea that percolation models are zero-temperature polymers. Remark 2.1 below explains.) For the RWRE we identify a family of stationary and ergodic distributions for the environment as seen from the particle. The averaged stationary RWRE is a standard random walk.

(ii) Logarithms of the limiting point-to-point ratios give us an analogue of Busemann functions in the positive temperature setting. Busemann functions have emerged as a central object in the study of geodesics and invariant distributions of percolation models and related interacting particle systems [1, 4, 11, 16, 21]. Our paper introduces this notion in the positive temperature setting. We show how Busemann functions solve a variational problem that characterizes the limiting free energy density of the log-gamma polymer.

A theme that appears more than once is a familiar large deviations duality between the asymptotic velocity of the path under polymer distributions and a tilt introduced into the partition function and probability distribution. In this duality the mapping from velocity to tilt is given by the expectation of the Busemann function. In particular, this duality determines how limits of ratios of point-to-point and tilted point-to-line partition functions match up with each other.

A word of explanation about our focus on the log-gamma polymer. The ultimate goal is of course to find results valid for a wide class of polymer models. We could formulate at least some of our results more generally. But the statements would be complicated and need hypotheses that we can presently verify only for the log-gamma model anyway. For general polymers, just as for general percolation models, we cannot currently prove even mild regularity properties for the limiting free energy. Thus we chose to focus exclusively on the log-gamma model (except for the general discussions in Sections 2 and 5).

We expect that much of this picture can eventually be verified for general $1+1$ dimensional directed polymers. Our hope is that this paper would inspire such further work. For example, it is clear that the solution of the variational formula for the free energy in terms of Busemann functions works completely generally, once a sufficiently strong existence statement for Busemann functions is proved. Busemann functions with tractable distributions are an essential feature of the exact solvability of the log-gamma polymer. They can be used to construct a shift-invariant version of the polymer model, which was earlier used for deriving fluctuation exponents and large deviation rate functions [15, 28].

The log-gamma polymer was introduced in [28] and subsequently linked with integrable systems and interesting combinatorics [2, 10, 22]. The log-gamma polymer is a canonical model in the Kardar–Parisi–Zhang universality class, in the same vein as the asymmetric simple exclusion process, the corner growth model with geometric or exponential weights and the semidiscrete polymer of O'Connell–Yor [9, 17, 23, 24, 29, 30]. These exactly solvable models are believed to be representative of what should be true more generally.

Organization of the paper. The paper is essentially self-contained. One exception is that in Section 5 we cite variational formulas for the free energy from [25, 26]. Here is an outline of the paper:

Section 2. Introduction of the polymer RWRE in a general context as the positive temperature counterpart of the competition interface of last-passage percolation.

Section 3. Introduction of the log-gamma polymer. The shift-invariant log-gamma polymer is formalized in the definition of a gamma system of weights.

Section 4. Limits of ratios of point-to-point partition functions for the log-gamma polymer.

Section 5. Busemann functions are constructed from limits of ratios of point-to-point partition functions and used to solve a variational formula for the limiting free energy. Duality between tilt and velocity.

Section 6. Limits of ratios of tilted point-to-line partition functions for the log-gamma polymer. Duality between tilt and velocity appears again.

Section 7. Limits of ratios of partition functions yield convergence of polymer measures to the polymer RWRE. The limit RWRE has fluctuations of size $n^{2/3}$ under the averaged measure.

Section 8. A stationary, ergodic distribution for the log-gamma polymer RWRE.

Section 8. Several auxiliary results, including a large deviation bound for the log-gamma polymer and a general ergodic theorem for cocycles.

Notation and conventions. $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$, $[n] = \{1, 2, \dots, n\}$. $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. On $\mathbb{R}^2$ the ℓ^1 norm is $|x|_1 = |x_1| + |x_2|$, the inner product is $x \cdot y = x_1 y_1 + x_2 y_2$, and inequalities are coordinatewise: $(x_1, x_2) \le (y_1, y_2)$ if $x_r \le y_r$ for $r \in \{1, 2\}$. Standard basis vectors are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Our random walks live in $\mathbb{Z}_+^2$ and admissible paths $x_\cdot = (x_k)_{k=0}^n$ have steps $z_k = x_k - x_{k-1} \in \mathcal{R} = \{e_1, e_2\}$. Points of $\mathbb{Z}_+^2$ are written as u, v, x, y but also as (m, n) or (i, j). Weights indexed by a single point do not have the parentheses: if $x = (i, j)$, then $\eta_x = \eta_{i,j}$. For $u \le v$ in $\mathbb{Z}_+^2$, $\Pi_{u,v}$ is the set of admissible paths from $x_0 = u$ to $x_{|v-u|_1} = v$. Limit velocities of these walks lie in the simplex $\mathcal{U} = \{(u, 1-u) : u \in [0, 1]\}$, whose (relative) interior is denoted by $\operatorname{int}\mathcal{U}$. Shift maps T_v act on suitably indexed configurations $w = (w_x)$ by $(T_v w)_x = w_{v+x}$. $\mathbb{E}$ and $\mathbb{P}$ refer to the random weights (the environment), and otherwise E^μ denotes expectation under probability measure μ. The usual gamma function for $\rho > 0$ is $\Gamma(\rho) = \int_0^\infty x^{\rho-1} e^{-x}\,dx$, and the Gamma($\rho$) distribution on $\mathbb{R}_+$ is $\Gamma(\rho)^{-1}x^{\rho-1}e^{-x}\,dx$. $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_1 = \Psi_0'$ are the digamma and trigamma functions.

The reader should be warned that several different partition functions appear in this paper. They are all denoted by Z and sometimes with additional notation such as $\check{Z}$. It should be clear from the context which Z is meant. Each Z is a sum of weights $W(x.)$ of paths $x.$ from a collection of nearest-neighbor lattice paths. Associated to each Z is a polymer probability measure Q on paths, $Q\{x.\} = Z^{-1}W(x.)$.

2. The polymer random walk in random environment. In this section we introduce a random walk in random environment (RWRE) associated to an underlying directed polymer model. This walk appears when we look for the positive temperature counterparts of the notions of geodesics and competition interface that appear in last-passage percolation. Percolation and polymers are discussed in this section in terms of real weights, without specifying probability distributions.

2.1. *Geodesics and competition interface in last-passage percolation.* We give a quick definition of last-passage percolation, also known as the zero-temperature polymer. Let $\{\omega_x : x \in \mathbb{Z}_+^2\}$ be a collection of real-valued weights. For $u \le v$ in $\mathbb{Z}_+^2$ let $\Pi_{u,v}$ denote the set of admissible lattice paths $x. = (x_i)_{0\le i\le n}$ with $n = |v-u|_1$ that satisfy $x_0 = u$, $x_i - x_{i-1} \in \{e_1, e_2\}$, $x_n = v$. The last-passage times are defined by

$$G_{u,v} = \max_{x.\in\Pi_{u,v}} \sum_{i=1}^{|v-u|_1} \omega_{x_i}, \qquad u \le v \text{ in } \mathbb{Z}_+^2.$$

A finite path $(x_i)_{0\le i\le n}$ in $\Pi_{u,v}$ is a *geodesic* between u and v if it is the maximizing path that realizes $G_{u,v}$, namely, $G_{u,v} = \sum_{i=1}^{n} \omega_{x_i}$. Every subpath of a geodesic is also a geodesic. Let us assume that no two paths of any length have equal sum of weights so that maximizing paths are unique. This would almost surely be the case, for example, if the weights are i.i.d. with a continuous distribution.

It is convenient to construct the geodesic from u to v backward, utilizing the iteration

$$G_{u,x} = G_{u,x-e_1} \vee G_{u,x-e_2} + \omega_x.$$

Start the construction with $x_n = v$. Suppose the segment $(x_k, x_{k+1}, \dots, x_n)$ of the geodesic has been constructed. If $x_k > u$ coordinatewise, set

$$x_{k-1} = \begin{cases} x_k - e_1, & \text{if } G_{u,x_k-e_1} > G_{u,x_k-e_2}, \\ x_k - e_2, & \text{if } G_{u,x_k-e_1} < G_{u,x_k-e_2}. \end{cases} \tag{2.1}$$

If $x_k \cdot e_r = u \cdot e_r$ for either $r = 1$ or $r = 2$, then define the remaining segment as $(x_0, \dots, x_k) = (u + ie_{3-r})_{0\le i\le k}$.

For a fixed initial point $u \in \mathbb{Z}_+^2$, the *geodesic spanning tree* $\mathcal{T}_u$ of the lattice $u + \mathbb{Z}_+^2$ is the union of all the geodesics from u to v, $v \in u + \mathbb{Z}_+^2$.

The *competition interface* $\varphi = (\varphi_k)_{k \in \mathbb{Z}_+}$ is a lattice path on $\mathbb{Z}_+^2$ defined as a function of $\{G_{0,v}\}_{v \in \mathbb{Z}_+^2}$. It starts at $\varphi_0 = 0$ and then chooses, at each step, the minimal G-value,

$$\varphi_{k+1} = \begin{cases} \varphi_k + e_1, & \text{if } G_{0,\varphi_k+e_1} < G_{0,\varphi_k+e_2}, \\ \varphi_k + e_2, & \text{if } G_{0,\varphi_k+e_1} > G_{0,\varphi_k+e_2}. \end{cases} \tag{2.2}$$

The relationship between $\mathcal{T}_0$ and φ is that φ separates the two subtrees $\mathcal{T}_{0,e_1}, \mathcal{T}_{0,e_2}$ of $\mathcal{T}_0$ rooted at e_1 and e_2. Since every $\mathbb{Z}_+^2$ lattice path from 0 has to go through either e_1 or e_2, $\mathcal{T}_0 = \{0\} \cup \mathcal{T}_{0,e_1} \cup \mathcal{T}_{0,e_2}$ as a disjoint union. For each $n \in \mathbb{Z}_+$, φ_n is the unique point such that $|\varphi_n|_1 = n$ and for $r \in \{1, 2\}$, $\{\varphi_n + ke_r : k \in \mathbb{N}\} \subseteq \mathcal{T}_{0,e_r}$. Note that we cannot say which tree contains φ_n, unless we know that $\varphi_n - \varphi_{n-1} = e_r$ in which case $\varphi_n \in \mathcal{T}_{0,e_r}$. If we shift φ by $(1/2, 1/2)$, then it threads exactly between the two trees (Figure 1).

The term competition interface comes from the interpretation that $\mathcal{T}_{0,e_1}$ and $\mathcal{T}_{0,e_2}$ are two competing clusters or infections on the lattice [13, 14]. The model can be defined dynamically. The clusters at time $t \in \mathbb{R}_+$ are $\mathcal{T}_{0,e_r}(t) = \{v \in \mathcal{T}_{0,e_r} : G_{0,v} \le t\}$.

2.2. *Geodesics and competition interface for a positive temperature polymer.* Let $\{V_x\}_{x \in \mathbb{Z}_+^2}$ be positive weights. Define point-to-point polymer partition functions for $u \le v$ in $\mathbb{Z}_+^2$ by

$$Z_{u,v} = \sum_{x_\cdot \in \Pi_{u,v}} \prod_{i=1}^{|v-u|_1} V_{x_i}^{-1} \tag{2.3}$$

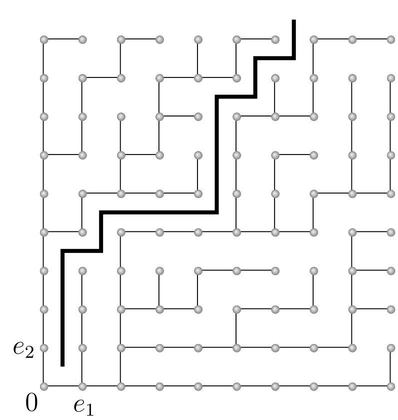

FIG. 1. *The competition interface shifted by* $(1/2, 1/2)$ (*solid line*) *separating the subtrees of* $\mathcal{T}_0$ *rooted at* e_1 *and* e_2.

and the polymer measure on the set of paths $\Pi_{u,v}$ by

$$Q_{u,v}\{x_\cdot\} = \frac{1}{Z_{u,v}} \prod_{i=1}^{|v-u|_1} V_{x_i}^{-1}, \qquad x_\cdot \in \Pi_{u,v}. \tag{2.4}$$

Our convention is to use reciprocals V_x^{-1} of the weights in the definitions. The reason is that this way the weights in the log-gamma polymer are gamma distributed and features of the beta-gamma algebra arise naturally.

REMARK 2.1. A conventional way of defining polymer partition functions is

$$Z_{u,v}^{\beta} = \sum_{x_\cdot \in \Pi_{u,v}} e^{\beta \sum_{i=1}^{|v-u|_1} \omega_{x_i}}$$

with an inverse temperature parameter $0 < \beta < \infty$. In the zero-temperature limit $\beta^{-1} \log Z_{u,v}^{\beta} \to G_{u,v}$ as $\beta \to \infty$, and the polymer measure $Q_{u,v}^{\beta}$ concentrates on the geodesic(s) from u to v. This is the sense in which last-passage percolation is the zero-temperature polymer. See Remark 3.2 below for this point for the log-gamma polymer.

We implement noisy versions of rules (2.1) and (2.2) to define positive temperature counterparts of geodesics and the competition interface.

Fix a base point $u \in \mathbb{Z}_+^2$ and define a backward Markov transition kernel $\overleftarrow{\pi}^u$ on the lattice $u + \mathbb{Z}_+^2$ by $\overleftarrow{\pi}^u(u,u) = 1$, and

$$\overleftarrow{\pi}^u(x, x - e_r) = \frac{V_x^{-1} Z_{u,x-e_r}}{Z_{u,x}} = \frac{Z_{u,x-e_r}}{Z_{u,x-e_1} + Z_{u,x-e_2}} \qquad \text{for } r \in \{1,2\}, \tag{2.5}$$

if both x and $x - e_r$ lie in $u + \mathbb{Z}_+^2$. The middle formula above gives the correct values on the boundaries of $u + \mathbb{Z}_+^2$ where there is only one admissible backward step, $\overleftarrow{\pi}^u(u + ie_r, u + (i-1)e_r) = 1$ for $i \geq 1$ and $r \in \{1,2\}$.

For a path $x_\cdot \in \Pi_{u,v}$ comparison of (2.4) and (2.5) shows

$$Q_{u,v}\{x_\cdot\} = \prod_{i=1}^{|v-u|_1} \overleftarrow{\pi}^u(x_i, x_{i-1}).$$

So the quenched polymer distribution $Q_{u,v}$ is the distribution of the backward Markov chain with initial state v, transition $\overleftarrow{\pi}^u$, and absorption at u. The distributions $Q_{u,v}$ are the noisy counterparts of geodesics. The nesting property of geodesics manifests itself through conditioning. Let $u < z < w < v$ in $\mathbb{Z}_+^2$. Let $A_{z,w}$ be the set of paths in $\Pi_{u,v}$ that go through the points z and w. Given $y_\cdot \in \Pi_{z,w}$, let $B_{y_\cdot}$ be the set of paths in $\Pi_{u,v}$ that traverse the path $y_\cdot$ (i.e., contain $y_\cdot$ as a subpath). Then

$$Q_{u,v}(B_{y_\cdot} | A_{z,w}) = Q_{z,w}\{y_\cdot\}.$$

Define the random geodesic spanning tree $\mathcal{T}_u$ rooted at u by choosing, for each $x \in (u + \mathbb{Z}_+^2) \setminus \{u\}$, a parent

$$\gamma(x) = \begin{cases} x - e_1, & \text{with probability } \overleftarrow{\pi}^u(x, x - e_1), \\ x - e_2, & \text{with probability } \overleftarrow{\pi}^u(x, x - e_2). \end{cases} \tag{2.6}$$

Now that we have the positive temperature counterparts of geodesics, we can find the positive temperature counterpart of the competition interface by reference to the tree $\mathcal{T}_0$ rooted at 0. Let $\mathcal{T}_{0,e_r}$ be the subtree rooted at e_r, so that $\mathcal{T}_0 = \{0\} \cup \mathcal{T}_{0,e_1} \cup \mathcal{T}_{0,e_2}$ as a disjoint union. The lemma below shows that there is a well-defined path $X_\cdot$ that separates the trees $\mathcal{T}_{0,e_1}$ and $\mathcal{T}_{0,e_2}$, and evolves as a Markov chain in the environment defined by the partition functions. In other words, this random walk in a random environment (RWRE) is the positive temperature analogue of the competition interface. The picture for $X_\cdot$ is the same as for φ in Figure 1.

LEMMA 2.2. (a) *Given the choices made in* (2.6), *there is a unique lattice path* $(X_n)_{n \in \mathbb{Z}_+}$ *with these properties*: $X_0 = 0$, $X_n - X_{n-1} \in \{e_1, e_2\}$, *and for each* n *and* $r \in \{1, 2\}$, $\{X_n + ke_r : k \in \mathbb{N}\} \subseteq \mathcal{T}_{0,e_r}$.

(b) X_n *is a Markov chain with transition matrix*

$$\pi_{x,x+e_r} = \frac{Z_{0,x+e_r}^{-1}}{Z_{0,x+e_1}^{-1} + Z_{0,x+e_2}^{-1}}, \qquad x \in \mathbb{Z}_+^2, r \in \{1, 2\}. \tag{2.7}$$

PROOF. (a) To prove the existence of the path, start with $X_0 = 0$, and iterate the following move: if $\gamma(X_n + e_1 + e_2) = X_n + e_r$, set $X_{n+1} = X_n + e_{3-r}$.

(b) Given the path $(X_k)_{k=0}^n$ with $X_n = x$, we choose $X_{n+1} = x + e_1$ if $\gamma(x + e_1 + e_2) = x + e_2$ which happens with probability

$$\frac{Z_{0,x+e_2}}{Z_{0,x+e_1} + Z_{0,x+e_2}} = \frac{Z_{0,x+e_1}^{-1}}{Z_{0,x+e_1}^{-1} + Z_{0,x+e_2}^{-1}}$$

and similarly for $X_{n+1} = x + e_2$ with the complementary probability. $\square$

A genuine RWRE transition probability satisfies $\pi_{x,y}(\omega) = \pi_{0,y-x}(T_x\omega)$ for shift mappings $(T_x)_{x \in \mathbb{Z}_+^2}$ acting on the environments ω. We augment the space of weights to achieve this. We need to be precise about the sets of sites on which various classes of weights are defined.

DEFINITION 2.3. A collection of positive real weights

$$(\xi, \eta, \zeta, \check{\xi}) = \{\xi_x, \eta_{x-e_2}, \zeta_{x-e_1}, \check{\xi}_{x-e_1-e_2} : x \in \mathbb{N}^2\}$$

satisfies *north–east* (*NE*) *induction* if these equations hold for each $x \in \mathbb{N}^2$

$$\eta_x = \xi_x \frac{\eta_{x-e_2}}{\eta_{x-e_2} + \zeta_{x-e_1}}, \qquad \zeta_x = \xi_x \frac{\zeta_{x-e_1}}{\eta_{x-e_2} + \zeta_{x-e_1}} \quad \text{and} \tag{2.8}$$

$$\check{\xi}_{x-e_1-e_2} = \eta_{x-e_2} + \zeta_{x-e_1}. \tag{2.9}$$

North–east induction simply keeps track of ratios of partition functions. Take as given the subcollection of weights $\{\xi_{i,j}, \eta_{i,0}, \zeta_{0,j} : i, j \in \mathbb{N}\}$ on $\mathbb{Z}_+^2$, and construct the polymer partition functions

$$Z_{0,v} = \sum_{x_\cdot \in \Pi_{0,v}} \prod_{i=1}^{|v|_1} V_{x_i}^{-1} \qquad \text{with } V_{i,j} = \begin{cases} \xi_{i,j}, & (i,j) \in \mathbb{N}^2, \\ \eta_{i,0}, & i \in \mathbb{N}, j = 0, \\ \zeta_{0,j}, & i = 0, j \in \mathbb{N}. \end{cases} \tag{2.10}$$

Then define

$$\eta_x = \frac{Z_{0,x-e_1}}{Z_{0,x}} \quad \text{and} \quad \zeta_x = \frac{Z_{0,x-e_2}}{Z_{0,x}} \qquad \text{for } x \in \mathbb{N}^2. \tag{2.11}$$

Now the subsystem (ξ, η, ζ) satisfies (2.8), as can be verified by induction. To get the full system $(\xi, \eta, \zeta, \check{\xi})$ just define $\check{\xi}_x = \eta_{x+e_1} + \zeta_{x+e_2}$ for $x \in \mathbb{Z}_+^2$.

Note that (2.11) is valid also on the boundaries, by the definition (2.10) of Z_{0,ke_r} for $k \in \mathbb{N}$. The reason for the distinct notation $\{\eta_{i,0}, \zeta_{0,j}\}$ for boundary weights in (2.10) is that these are also ratios of partition functions, just as η_x and ζ_x in (2.11). We shall find that in the interesting log-gamma models, the boundary weights $\{\eta_{i,0}, \zeta_{0,j}\}$ are different from the bulk weights $\{\xi_{i,j}\}$. The role of the $\check{\xi}$ weights is not clear yet, but they will become central in the log-gamma context.

Define the space of environments

$$\begin{aligned} \Omega_{\mathrm{NE}} = \bigl\{\omega = (\xi, \eta, \zeta, \check{\xi}) \in \mathbb{R}_+^{\mathbb{N}^2 + (\mathbb{N}\times\mathbb{Z}_+) + (\mathbb{Z}_+\times\mathbb{N}) + \mathbb{Z}_+^2} &: \\ (\xi, \eta, \zeta, \check{\xi}) \text{ satisfies NE induction}&\bigr\}. \end{aligned} \tag{2.12}$$

Translations act via $T_z\omega = (\xi_{z+\mathbb{N}^2}, \eta_{z+\mathbb{N}\times\mathbb{Z}_+}, \zeta_{z+\mathbb{Z}_+\times\mathbb{N}}, \check{\xi}_{z+\mathbb{Z}_+^2})$ for $z \in \mathbb{Z}_+^2$, where we introduced notation $\xi_{z+\mathbb{N}^2} = \{\xi_{z+x}\}_{x\in\mathbb{N}^2}$, and similarly for the other configurations.

DEFINITION 2.4. The *polymer random walk in random environment* is a RWRE with environment space Ω_{NE} and transition probability

$$\pi_{x,x+e_1}(\omega) = \frac{\eta_{x+e_1}}{\eta_{x+e_1} + \zeta_{x+e_2}} \quad \text{and} \quad \pi_{x,x+e_2}(\omega) = \frac{\zeta_{x+e_2}}{\eta_{x+e_1} + \zeta_{x+e_2}}. \tag{2.13}$$

This definition is the same as (2.7) with partition functions (2.10). The quenched path probabilities P^ω of this RWRE started at $x_0 = 0$ are defined by

$$P^\omega(X_0 = 0, X_1 = x_1, \dots, X_n = x_n) = \prod_{k=1}^n \pi_{x_{k-1},x_k}(\omega). \tag{2.14}$$

Distributions of X_n are again related to polymer distributions and partition functions. Define

$$\check{Z}_{0,v} = \sum_{x_\cdot \in \Pi_{0,v}} \prod_{i=0}^{|v|_1 - 1} \check{\xi}_{x_i}^{-1}, \qquad v \in \mathbb{Z}_+^2.$$

In contrast with (2.10), this time the weight at the origin is included but the weight at v excluded. From (2.9) and (2.14) we derive two formulas. First, the distribution of X_n is a ratio of partition functions

$$P^\omega(X_n = x) = \frac{\check{Z}_{0,x}}{Z_{0,x}} \qquad \text{for } x \in \mathbb{Z}_+^2 \text{ such that } |x|_1 = n.$$

Then if the walk is conditioned to go through a point, the distribution of the path segment is the polymer probability in $\check{\xi}$ weights: for $x_\cdot = (x_k)_{k=0}^n \in \Pi_{0,x_n}$,

$$\begin{aligned} &P^\omega(X_0 = 0, X_1 = x_1, \dots, X_n = x_n | X_n = x_n) \\ &\quad = \frac{1}{\check{Z}_{0,x_n}} \prod_{i=0}^{n-1} \check{\xi}_{x_i}^{-1} = \check{Q}_{0,x_n}\{x_\cdot\}. \end{aligned}$$

3. The log-gamma polymer. This section gives a quick definition of the log-gamma polymer and its Burke property. Let $0 < \lambda < \rho < \infty$.

DEFINITION 3.1. A collection $(\xi, \eta, \zeta, \check{\xi}) = \{\xi_x, \eta_{x-e_2}, \zeta_{x-e_1}, \check{\xi}_{x-e_1-e_2} : x \in \mathbb{N}^2\}$ of positive real random variables is a *gamma system* of weights with parameters (λ, ρ) if the following three properties hold:

(a) NE induction (Definition 2.3) holds: for each $x \in \mathbb{N}^2$, almost surely,

$$\begin{aligned} &\eta_x = \xi_x \frac{\eta_{x-e_2}}{\eta_{x-e_2} + \zeta_{x-e_1}}, \qquad \zeta_x = \xi_x \frac{\zeta_{x-e_1}}{\eta_{x-e_2} + \zeta_{x-e_1}} \quad \text{and} \\ &\check{\xi}_{x-e_1-e_2} = \eta_{x-e_2} + \zeta_{x-e_1}. \end{aligned} \tag{3.1}$$

(b) The marginal distributions of the variables are

$$\eta_x \sim \text{Gamma}(\lambda), \qquad \zeta_x \sim \text{Gamma}(\rho - \lambda) \quad \text{and} \quad \xi_x, \check{\xi}_x \sim \text{Gamma}(\rho). \tag{3.2}$$

(c) The variables $\{\xi_{i,j}, \eta_{i,0}, \zeta_{0,j} : i, j \in \mathbb{N}\}$ are mutually independent.

A triple (ξ, η, ζ) is a gamma system with parameters (λ, ρ) if conditions (a)–(c) are satisfied without the conditions on $\check{\xi}$.

Note that the ξ-weights are defined only in the bulk $\mathbb{N}^2$, while the $\check{\xi}$-weights are defined also on the boundaries and at the origin. Variables η_x and ζ_x can be thought of as weights on the edges, while ξ_x and $\check{\xi}_x$ are weights on the vertices. Edge weights will also be denoted by

$$\tau_{x-e_1,x} = \eta_x \quad \text{and} \quad \tau_{x-e_2,x} = \zeta_x. \tag{3.3}$$

A natural way to think about equations (3.1) for a fixed x is as a mapping of triples (Figure 2),

$$(\xi_x, \eta_{x-e_2}, \zeta_{x-e_1}) \mapsto (\eta_x, \zeta_x, \check{\xi}_{x-e_1-e_2}). \tag{3.4}$$

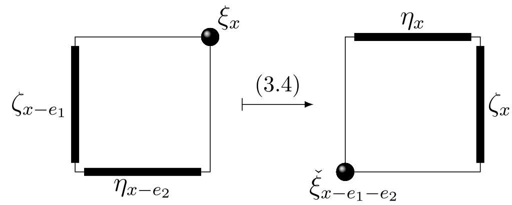

FIG. 2. *Mapping* (3.4) *that involves variables on a single lattice square. The picture illustrates how southwest corners are flipped into northeast corners in an inductive proof of the Burke property.*

This mapping has the property that if $(\xi_x, \eta_{x-e_2}, \zeta_{x-e_1})$ are independent with marginals (3.2), then the same is true for $(\eta_x, \zeta_x, \check{\xi}_{x-e_1-e_2})$, as can be checked, for example, via Laplace transforms. Consequently a gamma system $(\xi, \eta, \zeta, \check{\xi})$ can be constructed by repeated application of equations (3.1) to independent gamma variables given in (c).

An equivalent way to define the gamma system is to first construct the following polymer partition functions from the weights given in (c): for $0 \le u < v$ in $\mathbb{Z}_+^2$,

$$Z_{u,v} = \sum_{x_\cdot \in \Pi_{u,v}} \prod_{i=1}^{|v-u|_1} V_{x_i}^{-1} \qquad \text{with } V_{i,j} = \begin{cases} \xi_{i,j}, & (i,j) \in \mathbb{N}^2, \\ \eta_{i,0}, & i \in \mathbb{N}, j = 0, \\ \zeta_{0,j}, & i = 0, j \in \mathbb{N}. \end{cases} \tag{3.5}$$

Then, for $x \in \mathbb{Z}_+^2$ and $r \in \{1, 2\}$ such that $x - e_r \in \mathbb{Z}_+^2$, define

$$\tau_{x-e_r,x} = \frac{Z_{0,x-e_r}}{Z_{0,x}}.$$

The weights η_x and ζ_x are then defined via (3.3). Now we have a gamma system (ξ, η, ζ), which can be augmented to a gamma system $(\xi, \eta, \zeta, \check{\xi})$ since $\check{\xi}$ is a function of (η, ζ).

Mapping (3.4) furnishes the induction step in the proof of the *Burke property* of the log-gamma polymer ([28], Theorem 3.3): for any down-right path on $\mathbb{Z}_+^2$, the τ-variables on the path, the ξ variables strictly to the northeast of the path, and the $\check{\xi}$ variables strictly to the southwest of the path are all mutually independent with marginal distributions (3.2). The induction proof begins with the path that consists of the e_1- and e_2-axes. Southwest corners of the path can be flipped into northeast corners by an application of (3.4), as illustrated in Figure 2.

As an application of the Burke property, consider the down-right path consisting of the north and east boundaries of the rectangle $\{0, \dots, m\} \times \{0, \dots, n\}$. Then the Burke property gives us this statement:

$$\begin{gathered} \text{variables } \{\eta_{i,n}, \zeta_{m,j}, \check{\xi}_{i-1,j-1} : 1 \le i \le m, 1 \le j \le n\} \\ \text{are mutually independent with marginals (3.2).} \end{gathered} \tag{3.6}$$

REMARK 3.2. Let us revisit the zero temperature limit (Remark 2.1). The log-gamma polymer does not have an explicit β parameter, but ρ represents temperature. Replace ρ by $\varepsilon\rho$ in the definitions above, so that $\xi_x \sim \text{Gamma}(\varepsilon\rho)$. Then as $\varepsilon \searrow 0$, $-\varepsilon \log \xi_x \Rightarrow \omega_x$, a rate ρ exponential weight. For $u < v$ in $\mathbb{N}$,

$$\begin{aligned} \varepsilon \log Z_{u,v} &= \varepsilon \log \sum_{x_\cdot \in \Pi_{u,v}} \exp\left\{-\varepsilon^{-1} \sum_{i=1}^{|v-u|_1} \varepsilon \log \xi_{x_i}\right\} \\ &\Rightarrow \max_{x_\cdot \in \Pi_{u,v}} \sum_{i=1}^{|v-u|_1} \omega_x = G_{u,v} \qquad \text{as } \varepsilon \searrow 0. \end{aligned}$$

In other words, we have convergence in distribution to last-passage percolation with exponential weights.

An important function of the polymer path is the exit point or exit time t_{exit} of the path from the boundary: $t_{\text{exit}} = t_{e_1} \vee t_{e_2}$,

$$t_{e_1} = \max\{k \ge 0 : x_i = (i, 0) \text{ for } 0 \le i \le k\} \tag{3.7}$$

and

$$t_{e_2} = \max\{\ell \ge 0 : x_j = (0, j) \text{ for } 0 \le j \le \ell\}. \tag{3.8}$$

Note that for each path $t_{e_1} \wedge t_{e_2} = 0$. Partition functions (3.5) based at 0 can be equivalently written as

$$Z_{0,v} = \sum_{x_\cdot \in \Pi_{0,v}} \left(\prod_{i=1}^{t_{\text{exit}}} \tau_{x_{i-1},x_i}^{-1}\right)\left(\prod_{j=t_{\text{exit}}+1}^{|v|_1} \xi_{x_j}^{-1}\right), \qquad v \in \mathbb{Z}_+^2. \tag{3.9}$$

In a (λ, ρ) gamma system we have the means $\mathbb{E}(\log \eta_{i,0}) = \Psi_0(\lambda)$ and

$$\mathbb{E}(\log Z_{0,(m,n)}) = -m\Psi_0(\lambda) - n\Psi_0(\rho - \lambda). \tag{3.10}$$

The second one comes from

$$\log Z_{0,(m,n)} = -\sum_{i=1}^{m} \log \eta_{i,0} - \sum_{j=1}^{n} \log \zeta_{m,j}, \tag{3.11}$$

a sum of two correlated sums of i.i.d. random variables. Above $\Psi_0 = \Gamma'/\Gamma$ is the digamma function. It is strictly increasing on $(0, \infty)$, with $\Psi_0(0+) = -\infty$ and $\Psi_0(\infty) = \infty$. Its derivative is the trigamma function $\Psi_1 = \Psi_0'$ that is convex, strictly decreasing, with $\Psi_1(0+) = \infty$ and $\Psi_1(\infty) = 0$.

The asymptotic directions (or velocities) of admissible paths in $\mathbb{Z}_+^2$ lie in the simplex $\mathcal{U} = \{\mathbf{u} = (u, 1-u) : u \in [0, 1]\}$. Fundamental for the behavior of the log-gamma polymer is a 1–1 correspondence between velocities $\mathbf{u} \in \mathcal{U}$ and parameters $\lambda \in [0, \rho]$, for a fixed ρ. The *characteristic direction* for (λ, ρ) is

$$\mathbf{u}_{\lambda,\rho} = \left(\frac{\Psi_1(\rho - \lambda)}{\Psi_1(\lambda) + \Psi_1(\rho - \lambda)}, \frac{\Psi_1(\lambda)}{\Psi_1(\lambda) + \Psi_1(\rho - \lambda)}\right) \in \mathcal{U}. \tag{3.12}$$

Conversely, for $\mathbf{u} = (u, 1 - u)$, the unique parameter $\theta(u) = \theta(\mathbf{u}) \in [0, \rho]$ for which $\mathbf{u}$ is the characteristic direction is defined by $\theta(0) = 0$, $\theta(1) = \rho$ and

$$-u\Psi_1\bigl(\theta(u)\bigr) + (1 - u)\Psi_1\bigl(\rho - \theta(u)\bigr) = 0 \qquad \text{for } u \in (0, 1). \tag{3.13}$$

Function $\theta(u)$ is a strictly increasing bijective mapping between $u \in [0, 1]$ and $\theta \in [0, \rho]$.

The function $\theta(\mathbf{u})$ will appear throughout the paper. Let us point out that if $(m, n) = c\mathbf{u}$, then the right-hand side of (3.10) is minimized by $\lambda = \theta(\mathbf{u})$. As we shall see, this identifies the limiting free energy for the log-gamma polymer with i.i.d. Gamma(ρ) weights. Notationally, λ, α, ν denote generic parameters in $[0, \rho]$, while θ is reserved for the function defined above.

4. Limits of ratios of point-to-point partition functions. Fix $0 < \rho < \infty$. Let i.i.d. Gamma(ρ) weights $w = \{w_x : x \in \mathbb{Z}_+^2\}$ be given. Define partition functions

$$Z_{u,v} = \sum_{x_\cdot \in \Pi_{u,v}} \prod_{i=0}^{|v-u|_1 - 1} w_{x_i}^{-1}, \qquad 0 \le u \le v \text{ in } \mathbb{Z}_+^2. \tag{4.1}$$

Note that the weight at u is included and v excluded, in contrast with definitions (2.3) and (3.5). This is for convenience, to have clean limit statements below.

Suppose a lattice point $(m, n) \in \mathbb{N}^2$ tends to infinity in the first quadrant so that it has an asymptotic direction in the interior of the quadrant. Let $\lambda \in (0, \rho)$ be the unique value such that the following assumption holds:

$$m \wedge n \to \infty \quad \text{and} \quad \frac{m}{n} \to \frac{\Psi_1(\rho - \lambda)}{\Psi_1(\lambda)}. \tag{4.2}$$

When (4.2) holds we say that $(m, n) \to \infty$ in the characteristic direction of (λ, ρ).

The central theorem of this paper constructs gamma systems out of i.i.d. weights by taking limits of ratios of point-to-point partition functions.

THEOREM 4.1. *On the probability space of the i.i.d.* Gamma(ρ) *weights* $w = \{w_x : x \in \mathbb{Z}_+^2\}$, *there exist random variables* $\{\xi_x^\lambda, \eta_{x-e_2}^\lambda, \zeta_{x-e_1}^\lambda : \lambda \in (0, \rho), x \in \mathbb{N}^2\}$ *with the following properties*:

(i) *For each* $\lambda \in (0, \rho)$, $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$ *is a gamma system with parameters* (λ, ρ). *Furthermore, if on the same probability space there are additional random variables* $(\tilde\xi, \tilde\eta, \tilde\zeta) = \{\tilde\xi_x, \tilde\eta_{x-e_2}, \tilde\zeta_{x-e_1} : x \in \mathbb{N}^2\}$ *such that* $(\tilde\xi, \tilde\eta, \tilde\zeta, w)$ *is a gamma system with parameters* (ν, ρ), *then* $(\tilde\xi, \tilde\eta, \tilde\zeta) = (\xi^\nu, \eta^\nu, \zeta^\nu)$ *a.s.*

(ii) *Suppose a sequence* $(m, n) \to \infty$ *in the characteristic direction of* (λ, ρ), *as defined in* (4.2). *Then, for all* $x \in \mathbb{N} \times \mathbb{Z}_+$ *and* $y \in \mathbb{Z}_+ \times \mathbb{N}$, *these almost sure limits hold*:

$$\eta_x^\lambda = \lim_{(m,n)\to\infty} \frac{Z_{x,(m,n)}}{Z_{x-e_1,(m,n)}} \quad \textit{and} \quad \zeta_y^\lambda = \lim_{(m,n)\to\infty} \frac{Z_{y,(m,n)}}{Z_{y-e_2,(m,n)}} \tag{4.3}$$

and, furthermore, for all $1 \le p < \infty$,

$$\lim_{(m,n)\to\infty} \mathbb{E}\big[\big|\log Z_{x,(m,n)} - \log Z_{x-e_1,(m,n)} - \log \eta_x^\lambda\big|^p\big] = 0 \quad \textit{and}$$
$$\lim_{(m,n)\to\infty} \mathbb{E}\big[\big|\log Z_{y,(m,n)} - \log Z_{y-e_2,(m,n)} - \log \zeta_y^\lambda\big|^p\big] = 0. \tag{4.4}$$

(iii) *The weights are continuous in* λ, *and the edge weights are monotone in* λ: *for each* x *for which the weights are defined*, *almost surely*,

$$\eta_x^{\lambda_1} \le \eta_x^{\lambda_2} \quad \textit{and} \quad \zeta_x^{\lambda_1} \ge \zeta_x^{\lambda_2} \qquad \textit{for } \lambda_1 \le \lambda_2 \tag{4.5}$$

and

$$\eta_x^\lambda \to \eta_x^\nu, \qquad \zeta_x^\lambda \to \zeta_x^\nu, \qquad \xi_x^\lambda \to \xi_x^\nu \qquad \textit{as } \lambda \to \nu. \tag{4.6}$$

The rest of this section proves Theorem 4.1. The reader not interested in the (rather technical) proof can proceed to the next section where these limits are applied to solve a variational problem for the limiting free energy.

The proof relies on the following lemma for gamma systems. Let $(\xi, \eta, \zeta, \check{\xi})$ be an (α, ρ)-system according to Definition 3.1. Using the $\check{\xi}$ weights, define partition functions

$$\check{Z}_{u,v} = \sum_{x_\cdot \in \Pi_{u,v}} \prod_{i=0}^{|v-u|_1 - 1} (\check{\xi}_{x_i})^{-1}, \qquad 0 \le u \le v \text{ in } \mathbb{Z}_+^2, \tag{4.7}$$

and for $x \in \mathbb{N} \times \mathbb{Z}_+$ and $y \in \mathbb{Z}_+ \times \mathbb{N}$ edge ratio weights

$$\check{I}_{x,(m,n)} = \frac{\check{Z}_{x,(m,n)}}{\check{Z}_{x-e_1,(m,n)}} \quad \text{and} \quad \check{J}_{y,(m,n)} = \frac{\check{Z}_{y,(m,n)}}{\check{Z}_{y-e_2,(m,n)}}. \tag{4.8}$$

LEMMA 4.2. *Let* $0 < \lambda < \alpha < \tilde{\lambda} < \rho$. *Consider two sequences* $(m_i, n_i) \to \infty$ *and* $(\tilde{m}_j, \tilde{n}_j) \to \infty$ *in* $\mathbb{N}^2$ *such that*

$$\frac{m_i}{n_i} \to \frac{\Psi_1(\rho - \lambda)}{\Psi_1(\lambda)} \quad \textit{and} \quad \frac{\tilde{m}_j}{\tilde{n}_j} \to \frac{\Psi_1(\rho - \tilde{\lambda})}{\Psi_1(\tilde{\lambda})}.$$

Then for $x \in \mathbb{N} \times \mathbb{Z}_+$ *and* $y \in \mathbb{Z}_+ \times \mathbb{N}$,

$$\varlimsup_{i\to\infty} \check{I}_{x,(m_i,n_i)} \le \eta_x \le \varliminf_{j\to\infty} \check{I}_{x,(\tilde{m}_j,\tilde{n}_j)} \qquad \textit{a.s.} \tag{4.9}$$

and

$$\varlimsup_{j\to\infty} \check{J}_{y,(\tilde{m}_j,\tilde{n}_j)} \le \zeta_y \le \varliminf_{i\to\infty} \check{J}_{y,(m_i,n_i)} \qquad \textit{a.s.}$$

PROOF. For notational simplicity we drop the i, j indices from (m, n) and $(\tilde{m}, \tilde{n})$. We relate ratios (4.8) to ratios of partition functions with boundaries. Let $Z^{\mathrm{NE}}_{(k,\ell),(m,n)}$ denote a partition function that uses η and ζ weights on the north and east boundaries of the rectangle $\{k, \dots, m\} \times \{\ell, \dots, n\}$ and $\check{\xi}$ weights in the bulk:

$$Z^{\mathrm{NE}}_{(k,n),(m,n)} = \prod_{s=k+1}^{m} \frac{1}{\eta_{s,n}},$$

$$Z^{\mathrm{NE}}_{(m,\ell),(m,n)} = \prod_{t=\ell+1}^{n} \frac{1}{\zeta_{m,t}},$$

and for $0 \le k < m$ and $0 \le \ell < n$

$$\begin{aligned} Z^{\mathrm{NE}}_{(k,\ell),(m,n)} &= \sum_{i=k}^{m-1} \check{Z}_{(k,\ell),(i,n-1)} \frac{1}{\check{\xi}_{i,n-1}} \prod_{s=i+1}^{m} \frac{1}{\eta_{s,n}} \\ &\quad + \sum_{j=\ell}^{n-1} \check{Z}_{(k,\ell),(m-1,j)} \frac{1}{\check{\xi}_{m-1,j}} \prod_{t=j+1}^{n} \frac{1}{\zeta_{m,t}}. \end{aligned} \tag{4.10}$$

In the last formula $Z^{\mathrm{NE}}_{(k,\ell),(m,n)}$ is decomposed according to the entry points (i, n) and (m, j) of the paths on the north and east boundaries. If the entry is at (i, n), the first boundary variable encountered is $\eta_{i+1,n}$ associated to the edge $\{(i, n), (i + 1, n)\}$. The last bulk weight $\check{\xi}_{i,n-1}$ has to be inserted explicitly into the formula because $\check{Z}_{(k,\ell),(i,n-1)}$ does not include the weight at $(i, n - 1)$, by its definition (4.7).

The corresponding ratio weights on edges are

$$I_{(k,\ell),(m,n)} = \frac{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}}{Z^{\mathrm{NE}}_{(k-1,\ell),(m,n)}} \quad \text{and} \quad J_{(k,\ell),(m,n)} = \frac{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}}{Z^{\mathrm{NE}}_{(k,\ell-1),(m,n)}}. \tag{4.11}$$

Due to the reversibility of the shift-invariant setting, these ratio weights are the same as the original ratio weights, and thereby do not depend on (m, n). This is the content of the next lemma.

LEMMA 4.3. *For $0 \le k \le m$ and $0 \le \ell \le n$ such that the weights below are defined*,

$$\eta_{k,\ell} = I_{(k,\ell),(m,n)} \quad \textit{and} \quad \zeta_{k,\ell} = J_{(k,\ell),(m,n)}. \tag{4.12}$$

PROOF. When $\ell = n$ in the η-identity or $k = m$ in the ζ-identity, the claims follow from the definitions. Here is the induction step for $\eta_{k,\ell}$, assuming the identities have been verified for the edges $\{(k - 1, \ell + 1), (k, \ell + 1)\}$ and

$\{(k,\ell),(k,\ell+1)\}$, closest to the north and east of the edge $\{(k-1,\ell),(k,\ell)\}$:

$$\begin{aligned}
I_{(k,\ell),(m,n)} &= \frac{\check{\xi}_{k-1,\ell} Z^{\mathrm{NE}}_{(k,\ell),(m,n)}}{Z^{\mathrm{NE}}_{(k,\ell),(m,n)} + Z^{\mathrm{NE}}_{(k-1,\ell+1),(m,n)}} \\
&= \check{\xi}_{k-1,\ell}\left(1 + \frac{Z^{\mathrm{NE}}_{(k-1,\ell+1),(m,n)}}{Z^{\mathrm{NE}}_{(k,\ell+1),(m,n)}} \cdot \frac{Z^{\mathrm{NE}}_{(k,\ell+1),(m,n)}}{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}}\right)^{-1} \\
&= \check{\xi}_{k-1,\ell}\left(1 + \frac{\zeta_{k,\ell+1}}{\eta_{k,\ell+1}}\right)^{-1} = (\eta_{k,\ell} + \zeta_{k-1,\ell+1})\left(1 + \frac{\zeta_{k-1,\ell+1}}{\eta_{k,\ell}}\right)^{-1} \\
&= \eta_{k,\ell}.
\end{aligned}$$

The third equality is the induction step. The fourth equality uses (3.1) twice. $\square$

We need one more variant of ratio weights, namely the types where the last step of the path is restricted to either e_1 or e_2. Relative to any fixed rectangle $\{k,\dots,m\} \times \{\ell,\dots,n\}$, define the distances of the entrance points of the polymer path $x_\cdot \in \Pi_{(k,\ell),(m,n)}$ on the north and east boundaries to the corner (m,n),

$$t^*_{e_1} = \max\{r \ge 0 : x_{m-k+n-\ell-i} = (m-i,n) \text{ for } 0 \le i \le r\} \tag{4.13}$$

and

$$t^*_{e_2} = \max\{r \ge 0 : x_{m-k+n-\ell-j} = (m,n-j) \text{ for } 0 \le j \le r\}. \tag{4.14}$$

For a subset A of paths, write $Z(A)$ for the partition function of paths restricted to A (in other words, for the unnormalized polymer measure). Then define, for $r \in \{1,2\}$,

$$\begin{aligned}
I^{e_r}_{(k,\ell),(m,n)} &= \frac{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}(t^*_{e_r} > 0)}{Z^{\mathrm{NE}}_{(k-1,\ell),(m,n)}(t^*_{e_r} > 0)} \quad \text{and} \\
J^{e_r}_{(k,\ell),(m,n)} &= \frac{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}(t^*_{e_r} > 0)}{Z^{\mathrm{NE}}_{(k,\ell-1),(m,n)}(t^*_{e_r} > 0)}.
\end{aligned} \tag{4.15}$$

We are ready to prove Lemma 4.2. We go through the proof of (4.9), the case for $\check{J}$ being the same. Applying Lemma A.1 from the Appendix (to a reversed rectangle) gives

$$\begin{aligned}
\eta_{k,\ell} + \big(I^{e_1}_{(k,\ell),(m+1,n+1)} - \eta_{k,\ell}\big) &\le \check{I}_{(k,\ell),(m,n)} \\
&\le \eta_{k,\ell} + \big(I^{e_2}_{(k,\ell),(m+1,n+1)} - \eta_{k,\ell}\big).
\end{aligned} \tag{4.16}$$

Taking (4.12) into consideration, the task is

$$\begin{aligned}
&\overline{\lim_{(m,n)\to\infty}} \big\{I^{e_2}_{(k,\ell),(m+1,n+1)} - I_{(k,\ell),(m+1,n+1)}\big\} \\
&\qquad \le 0 \le \underline{\lim}_{(\tilde{m},\tilde{n})\to\infty} \big\{I^{e_1}_{(k,\ell),(\tilde{m}+1,\tilde{n}+1)} - I_{(k,\ell),(\tilde{m}+1,\tilde{n}+1)}\big\}.
\end{aligned} \tag{4.17}$$

We do the first limit for e_2. The second is similar. Introduce the parameter

$$N = \frac{m+n}{\Psi_1(\rho-\lambda)+\Psi_1(\lambda)} \to \infty \tag{4.18}$$

with the property that $(m,n)/N \to (\Psi_1(\rho-\lambda), \Psi_1(\lambda))$. The first inequality of (4.17) follows from showing that $\forall \varepsilon > 0\ \exists a > 0$ such that

$$\mathbb{P}\{I^{e_2}_{(k,\ell),(m+1,n+1)} \geq I_{(k,\ell),(m+1,n+1)} + \varepsilon\} \leq 2e^{-aN}. \tag{4.19}$$

Introduce the quenched path measure $Q^{\mathrm{NE}}_{(k,\ell),(m,n)}$ that corresponds to the partition function in (4.10):

$$\begin{aligned} I^{e_2}_{(k,\ell),(m+1,n+1)} &= \frac{Z^{\mathrm{NE}}_{(k,\ell),(m+1,n+1)}(t^*_{e_2} > 0)}{Q^{\mathrm{NE}}_{(k-1,\ell),(m+1,n+1)}(t^*_{e_2} > 0) \cdot Z^{\mathrm{NE}}_{(k-1,\ell),(m+1,n+1)}} \\ &\leq \frac{I_{(k,\ell),(m+1,n+1)}}{Q^{\mathrm{NE}}_{(k-1,\ell),(m+1,n+1)}(t^*_{e_2} > 0)}. \end{aligned}$$

For small enough ε_N the probability in (4.19) is bounded above by the sum

$$\mathbb{P}\left\{I_{(k,\ell),(m+1,n+1)} \geq \frac{\varepsilon}{2\varepsilon_N}\right\} + \mathbb{P}\{Q^{\mathrm{NE}}_{(k-1,\ell),(m+1,n+1)}(t^*_{e_1} > 0) \geq \varepsilon_N\}. \tag{4.20}$$

Note that the event in the Q^{NE}-probability was replaced by its complement. A sequence $0 < \varepsilon_N \searrow 0$ will be chosen below.

By (4.12) $I_{(k,\ell),(m+1,n+1)}$ has Gamma(α) distribution, and so the first probability in (4.20) is bounded by $e^{-c\varepsilon/\varepsilon_N}$.

We show that the Q^{NE}-probability in (4.20) is actually a large deviation by replacing (m,n) with a direction that is characteristic for (α,ρ). The next lemma contains the idea for replacing (m,n).

LEMMA 4.4. *Let $(\bar m, \bar n)$ satisfy $\bar m > m$ and $\ell < \bar n < n$. Then*

$$Q^{\mathrm{NE}}_{(k,\ell),(m,n)}(t^*_{e_1} > 0) = Q^{\mathrm{NE}}_{(k,\ell),(\bar m,\bar n)}(t^*_{e_1} > \bar m - m).$$

PROOF. A path in $\Pi_{(k,\ell),(m,n)}$ that satisfies $t^*_{e_1} > 0$ must use one of the edges $\{(i,\bar n - 1),(i,\bar n)\}$, $k \leq i \leq m-1$. Otherwise it hits the east boundary first and $t^*_{e_1} = 0$. Decomposing according to this choice of edge and using definition (4.10),

$$Q^{\mathrm{NE}}_{(k,\ell),(m,n)}(t^*_{e_1} > 0) = \sum_{i=k}^{m-1} \check Z_{(k,\ell),(i,\bar n-1)} \frac{1}{\check\xi_{i,\bar n-1}} \cdot \frac{Z^{\mathrm{NE}}_{(i,\bar n),(m,n)}}{Z^{\mathrm{NE}}_{(k,\ell),(m,n)}}.$$

By Lemma 4.3 the last ratio does not depend on (m,n), and (m,n) can be replaced by $(\bar m,\bar n)$. This moves the northeast corner in definition (4.10) to $(\bar m,\bar n)$, as well as the reference point of $t^*_{e_1}$ in (4.13). Since the sum still runs up to $m-1$, it now represents paths in $\Pi_{(k,\ell),(\bar m,\bar n)}$ that hit the north boundary to the left of $(m,\bar n)$. This proves Lemma 4.4. $\square$

Take

$$(\bar m,\bar n)=(\lfloor N\Psi_1(\rho-\alpha)\rfloor+k-1,\lfloor N\Psi_1(\alpha)\rfloor+\ell), \tag{4.21}$$

essentially the characteristic direction for (α,ρ). Since $\lambda<\alpha$ and Ψ_1 is strictly decreasing, there exists $\gamma>0$ such that for large enough N, $\bar m\ge m+1+N\gamma$ and $\bar n\le n-N\gamma$. Put $\varepsilon_N=e^{-\delta_1\gamma N}$ for a small enough $\delta_1>0$. Then for large enough N,

$$\begin{aligned}&\mathbb{P}\{Q^{\mathrm{NE}}_{(k-1,\ell),(m+1,n+1)}(t^*_{e_1}>0)\ge\varepsilon_N\}\\&\quad\le\mathbb{P}\{Q^{\mathrm{NE}}_{(k-1,\ell),(\bar m,\bar n)}(t^*_{e_1}>N\gamma)\ge e^{-\delta_1\gamma N}\}\le e^{-c_1\gamma N}.\end{aligned} \tag{4.22}$$

The last inequality came from Lemma A.2 in the Appendix where we can take $\kappa_N=1$ and $\delta\le\gamma$. Both probabilities in (4.20) have been shown to decay exponentially in N, and consequently (4.19) holds. This completes the proof of Lemma 4.2. $\square$

Turning to the proof of Theorem 4.1, we begin by showing the a.s. convergence in (4.3) for a fixed sequence and fixed λ. Later, when we finish the proof of Theorem 4.1, the comparisons of Lemma 4.2 allow us to extend the limit to all sequences with an asymptotic direction. Define ratio variables by

$$\eta_{x,(m,n)}=\frac{Z_{x,(m,n)}}{Z_{x-e_1,(m,n)}}\quad\text{and}\quad\zeta_{y,(m,n)}=\frac{Z_{y,(m,n)}}{Z_{y-e_2,(m,n)}} \tag{4.23}$$

for $x\in\mathbb{N}\times\mathbb{Z}_+$ and $y\in\mathbb{Z}_+\times\mathbb{N}$.

PROPOSITION 4.5. *Fix $0<\lambda<\rho$ and fix a sequence $(m,n)\to\infty$ as in (4.2). Then for all $x\in\mathbb{N}\times\mathbb{Z}_+$ and $y\in\mathbb{Z}_+\times\mathbb{N}$ the almost sure limits*

$$\eta_x=\lim_{(m,n)\to\infty}\eta_{x,(m,n)}\quad\text{and}\quad\zeta_y=\lim_{(m,n)\to\infty}\eta_{y,(m,n)} \tag{4.24}$$

exist and have distributions $\eta_x\sim\mathrm{Gamma}(\lambda)$ and $\zeta_y\sim\mathrm{Gamma}(\rho-\lambda)$.

PROOF. We treat the case of the η variables, the case for ζ being identical. For a while, until otherwise indicated, we are considering a fixed sequence of lattice points that satisfies $(m,n)\to\infty$ as in (4.2). To avoid extra notation we refrain from indexing the lattice points, as in (m_k,n_k). Later we can improve the result so that the limit only depends on λ and not on the particular sequence $(m,n)\to\infty$.

We show that for $0<s<\infty$ the distribution functions

$$\begin{aligned}G^*(s)&=\mathbb{P}\Big\{\overline{\lim_{(m,n)\to\infty}}\,\eta_{x,(m,n)}\le s\Big\}\quad\text{and}\\G_*(s)&=\mathbb{P}\Big\{\underline{\lim}_{(m,n)\to\infty}\,\eta_{x,(m,n)}\le s\Big\}\end{aligned} \tag{4.25}$$

satisfy $G^*(s) = G_*(s) = F_\lambda(s)$ where

$$F_\lambda(s) = \Gamma(\lambda)^{-1} \int_0^s t^{\lambda-1} e^{-t}\, dt$$

is the c.d.f. of the Gamma(λ) distribution. Since $\underline{\lim}\, \eta_{x,(m,n)} \le \overline{\lim}\, \eta_{x,(m,n)}$, this suffices for the conclusion. Working with the distributions allows us to use any particular construction of the processes.

Let $\{U_{i,j}\}$ be i.i.d. Uniform(0, 1) random variables. For $i, j \in \mathbb{N}$ and $\alpha \in (0, \rho)$ define

$$\bar\eta^{\alpha}_{i,0} = F_\alpha^{-1}(U_{i,0}) \quad \text{and} \quad \bar\zeta^{\alpha}_{0,j} = F_{\rho-\alpha}^{-1}(U_{0,j}). \tag{4.26}$$

This gives coupled weights $\bar\eta^\alpha_{i,0} \sim$ Gamma(α) on the south boundary and $\bar\zeta^\alpha_{0,j} \sim$ Gamma($\rho - \alpha$) on the west boundary of the positive quadrant. For the bulk weights take an i.i.d. collection $\{\sigma_x\}_{x\in\mathbb{N}^2}$ of Gamma(ρ) weights independent of $\{U_{i,j}\}$.

As mentioned after Definition 3.1, the mutually independent initial weights $\{\sigma_{i,j}, \bar\eta^\alpha_{i,0}, \bar\zeta^\alpha_{0,j} : i, j \in \mathbb{N}\}$ can be extended to the full gamma (α, ρ) system $(\sigma, \bar\eta^\alpha, \bar\zeta^\alpha, \check\sigma^{[\alpha]})$. The construction preserves monotonicity of the edge weights, so that

$$\bar\eta^\alpha_{i,j} \le \bar\eta^\nu_{i,j} \quad \text{and} \quad \bar\zeta^\alpha_{i,j} \ge \bar\zeta^\nu_{i,j} \qquad \text{for } \alpha \le \nu.$$

Superscript $[\alpha]$ reminds us that even though the variables $\{\check\sigma^{[\alpha]}_{i,j}\}_{i,j\ge 0}$ are i.i.d. Gamma(ρ) for each $\alpha \in (0, \rho)$, they were computed from α-boundary conditions. Define partition functions

$$\check Z^{[\alpha]}_{u,v} = \sum_{x_\cdot \in \Pi_{u,v}} \prod_{i=0}^{|v-u|_1 - 1} \big(\check\sigma^{[\alpha]}_{x_i}\big)^{-1}, \qquad 0 \le u \le v \text{ in } \mathbb{Z}^2, \tag{4.27}$$

and edge ratio weights

$$\check I^{[\alpha]}_{x,(m,n)} = \frac{\check Z^{[\alpha]}_{x,(m,n)}}{\check Z^{[\alpha]}_{x-e_1,(m,n)}} \quad \text{and} \quad \check J^{[\alpha]}_{y,(m,n)} = \frac{\check Z^{[\alpha]}_{y,(m,n)}}{\check Z^{[\alpha]}_{y-e_2,(m,n)}}. \tag{4.28}$$

For each $\alpha \in (0, \rho)$, we have equality in distribution of processes

$$\{\check I^{[\alpha]}_{(i+1,j),(m,n)}, \check J^{[\alpha]}_{(i,j+1),(m,n)}, \check\sigma^{[\alpha]}_{i,j}\} \overset{d}{=} \{\eta_{(i+1,j),(m,n)}, \zeta_{(i,j+1),(m,n)}, w_{i,j}\}. \tag{4.29}$$

These processes are indexed by $\{(i, j), (m, n) \in \mathbb{Z}^2_+ : (m, n) \ge (i + 1, j + 1)\}$. The equality in distribution comes from identical constructions applied to i.i.d. Gamma(ρ) weights: on the left to $\check\sigma^{[\alpha]}$, on the right to w. Now in (4.25) we can use any process $\{\check I^{[\alpha]}_{x,(m,n)}\}$.

For any $0 < \alpha_1 < \lambda < \alpha_2 < \rho$, applying Lemma 4.2 to two gamma systems $(\sigma, \bar\eta^{\alpha_1}, \bar\zeta^{\alpha_1}, \check\sigma^{[\alpha_1]})$ and $(\sigma, \bar\eta^{\alpha_2}, \bar\zeta^{\alpha_2}, \check\sigma^{[\alpha_2]})$ gives

$$\varliminf_{(m,n)\to\infty} \check I^{[\alpha_1]}_{(k,\ell),(m,n)} \ge \bar\eta^{\alpha_1}_{k,\ell} \quad \text{and} \quad \varlimsup_{(m,n)\to\infty} \check I^{[\alpha_2]}_{(k,\ell),(m,n)} \le \bar\eta^{\alpha_2}_{k,\ell} \qquad \text{a.s.} \tag{4.30}$$

By the equality in distribution (4.29),

$$G_*(s) = \mathbb{P}\Big\{ \varliminf_{(m,n)\to\infty} \check{I}^{[\alpha_1]}_{(k,\ell),(m,n)} \le s \Big\} \le \mathbb{P}\{\bar\eta^{\alpha_1}_{k,\ell} \le s\} = F_{\alpha_1}(s) \searrow F_\lambda(s)$$

as $\alpha_1 \nearrow \lambda$, and

$$G^*(s) = \mathbb{P}\Big\{ \varlimsup_{(m,n)\to\infty} \check{I}^{[\alpha_2]}_{(k,\ell),(m,n)} \le s \Big\} \ge F_{\alpha_2}(s) \nearrow F_\lambda(s) \qquad \text{as } \alpha_2 \searrow \lambda.$$

This gives $F_\lambda(s) \le G^*(s) \le G_*(s) \le F_\lambda(s)$ and completes the proof of Proposition 4.5. $\square$

Proposition 4.5 gave the a.s. convergence of ratios along a fixed sequence and for a given $\lambda \in (0, \rho)$. Next we construct a system of weights (ξ, η, ζ, w) from the limits (4.24) by defining

$$\xi_x = \eta_x + \zeta_x \qquad \text{for } x \in \mathbb{N}^2.$$

PROPOSITION 4.6. *The collection* (ξ, η, ζ, w) *is a gamma system with parameters* (λ, ρ), *that is*, *it satisfies Definition* 3.1.

PROOF. Equations (3.1) follow from the limits (4.24) and

$$w_x = \frac{Z_{x+e_1,(m,n)} + Z_{x+e_2,(m,n)}}{Z_{x,(m,n)}}.$$

By the equality in distribution in (4.29), it also follows that the limits in (4.30) exist,

$$\check{I}^{[\alpha]}_{k,\ell} = \lim_{(m,n)\to\infty} \check{I}^{[\alpha]}_{(k,\ell),(m,n)} \qquad \text{a.s.} \tag{4.31}$$

Let $0 < \alpha_1 < \lambda < \alpha_2 < \rho$. Utilizing (4.29), (4.30) and (4.31),

$$(\eta, w) \overset{d}{=} \big(\check{I}^{[\alpha_1]}, \check{\sigma}^{[\alpha_1]}\big) \ge \big(\bar\eta^{\alpha_1}, \check{\sigma}^{[\alpha_1]}\big) \underset{\alpha_1 \nearrow \lambda}{\longrightarrow} \big(\bar\eta^{\lambda}, \check{\sigma}^{[\lambda]}\big)$$

and

$$(\eta, w) \overset{d}{=} \big(\check{I}^{[\alpha_2]}, \check{\sigma}^{[\alpha_2]}\big) \le \big(\bar\eta^{\alpha_2}, \check{\sigma}^{[\alpha_2]}\big) \underset{\alpha_2 \searrow \lambda}{\longrightarrow} \big(\bar\eta^{\lambda}, \check{\sigma}^{[\lambda]}\big).$$

The inequalities and the convergence are a.s. and coordinatewise. The convergence follows from the continuity of definitions (4.26) in α and the continuity in equations (3.1) that inductively define the $(\bar\eta^\alpha, \bar\zeta^\alpha, \check{\sigma}^{[\alpha]})$ weights. The consequence is that

$$(\eta, w) \overset{d}{=} \big(\bar\eta^{\lambda}, \check{\sigma}^{[\lambda]}\big). \tag{4.32}$$

Equations $\zeta_x = w_{x-e_2} - \eta_{x-e_2+e_1}$ and $\xi_x = \eta_x + \zeta_x$ map (η, w) to the full system (ξ, η, ζ, w). The same mapping applied to the right-hand side of (4.32) recreates the system $(\sigma, \bar{\eta}^\lambda, \bar{\xi}^\lambda, \check{\sigma}^{[\lambda]})$, which we know to be a (λ, ρ) gamma system by its construction below (4.26). $\square$

PROOF OF THEOREM 4.1. Fix a countable dense subset D of $(0, \rho)$ and $\forall \lambda \in D$ a sequence $(m, n) \to \infty$ that satisfies (4.2). By Propositions 4.5 and 4.6, we can use limits (4.3) along these particular sequences to define, almost surely, (λ, ρ) gamma systems $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$ for $\lambda \in D$. Monotonicity (4.5) is satisfied a.s. for $\lambda_1, \lambda_2 \in D$ by Lemma 4.2. [The point is that the $\check{Z}$ partition functions in (4.7) are the same for all systems $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$.]

Monotonicity and known gamma distributions give also the limits in (4.6) when $\lambda \to \nu$ in D. For example, suppose $\lambda \nearrow \nu$ in D. Then $\lim_{\lambda \nearrow \nu} \eta_x^\lambda \le \eta_x^\nu$, but both are Gamma$(\nu)$ distributed and hence coincide a.s. The limit $\xi_x^\lambda \to \xi_x^\nu$ comes from the limits of η and ζ and $\xi_x^\lambda = \eta_x^\lambda + \zeta_x^\lambda$.

Extend the weights to all $\lambda \in (0, \rho)$ by defining

$$\eta_x^\lambda = \inf\{\eta_x^\nu : \nu \in D \cap (\lambda, \rho)\} = \sup\{\eta_x^\alpha : \alpha \in D \cap (0, \lambda)\} \tag{4.33}$$

with the obvious counterpart for ζ_x^λ and then $\xi_x^\lambda = \eta_x^\lambda + \zeta_x^\lambda$. The inf and the sup in (4.33) must agree a.s. because (i) the sup is not above the inf on account of the monotonicity for $\lambda \in D$, and (ii) they are both Gamma(λ) distributed. By the same reasoning, for $\lambda \in D$ definition (4.33) gives a.s. back the same value η_x^λ as originally constructed.

To check that the new system $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$ is a (λ, ρ) gamma system, fix a sequence $D \ni \alpha_i \nearrow \lambda$, and observe that equations (3.1) are preserved by limits, and the correct distributions come also through the limit. Extending properties (iii) utilizes monotonicity again. Limits (4.3) of ratios for arbitrary sequences, including for $\lambda \notin D$, come from the comparisons of Lemma 4.2 with the sequences fixed in the beginning of this proof.

The uniqueness in part (i) follows from Lemma 4.2 because the limits (4.3) imply that $\eta_x^{\alpha_1} \le \tilde{\eta}_x \le \eta_x^{\alpha_2}$ for all $\alpha_1 < \nu < \alpha_2$.

As the last item we prove the L^p convergence (4.4). Let $\eta_{x,(m,n)}$ and $\zeta_{y,(m,n)}$ be as in (4.23). It suffices to show that for each $p \in [1, \infty)$, there exists a finite constant $C(p)$ such that

$$\mathbb{E}\big[|\log \eta_{x,(m,n)}|^p\big] \le C(p) \qquad \text{for all } (m,n) \text{ in the sequence.} \tag{4.34}$$

The argument for $\zeta_{y,(m,n)}$ is analogous, or comes by transposition. The proof splits into separate bounds for plus and minus parts. The plus part is quick.

$$\frac{Z_{x,(m,n)}}{Z_{x-e_1,(m,n)}} = \frac{Z_{x,(m,n)}}{w_{x-e_1}^{-1}(Z_{x,(m,n)} + Z_{x-e_1+e_2,(m,n)})} \le w_{x-e_1}$$

from which, for all x, (m, n) and $1 \le p < \infty$,

$$\mathbb{E}\big[(\log^+ \eta_{x,(m,n)})^p\big] \le C(p) < \infty.$$

For the minus part, pick $\alpha \in (0, \lambda)$ and $\varepsilon > 0$. In the next derivation use distributional equality (4.29), bring in the ratio variables (4.11) and (4.15) with north–east boundaries with parameter α, and finally use the Schwarz inequality and $\check{I}^{[\alpha]}_{x,(m,n)} \geq I^{e_1}_{x,(m+1,n+1)}$ from (4.16):

$$
\begin{aligned}
&\mathbb{E}\big[\big(\log^- \eta_{x,(m,n)}\big)^p\big] \\
&\quad = \mathbb{E}\big[\big(\log^- \check{I}^{[\alpha]}_{x,(m,n)}\big)^p\big] \\
&\quad = \mathbb{E}\big[\big(\log^- \check{I}^{[\alpha]}_{x,(m,n)}\big)^p, I^{e_1}_{x,(m+1,n+1)} \leq (1-\varepsilon) I_{x,(m+1,n+1)}\big] \\
&\qquad + \mathbb{E}\Bigg[\bigg(\log \frac{1}{\check{I}^{[\alpha]}_{x,(m,n)}}\bigg)^p, \check{I}^{[\alpha]}_{x,(m,n)} \leq 1, \\
&\qquad\qquad I^{e_1}_{x,(m+1,n+1)} > (1-\varepsilon) I_{x,(m+1,n+1)}\Bigg]
\end{aligned}
$$

$$
\leq \big\{\mathbb{E}\big[\big(\log^- \check{I}^{[\alpha]}_{x,(m,n)}\big)^{2p}\big]\big\}^{1/2} \big\{\mathbb{P}\big(I^{e_1}_{x,(m+1,n+1)} \leq (1-\varepsilon) I_{x,(m+1,n+1)}\big)\big\}^{1/2} \tag{4.35}
$$

$$
+ \mathbb{E}\bigg[\bigg|\log \frac{1}{I_{x,(m+1,n+1)}}\bigg|^p\bigg] + \log \frac{1}{1-\varepsilon}. \tag{4.36}
$$

By Lemma 4.3 $I_{x,(m+1,n+1)}$ is a Gamma(α) variable, and consequently line (4.36) is a constant, independent of x and (m, n).

It remains to show that line (4.35) is bounded by a constant. From

$$
\begin{aligned}
I^{e_1}_{x,(m+1,n+1)} &= \frac{Z^{\mathrm{NE}}_{x,(m+1,n+1)}(t^*_{e_1} > 0)}{Z^{\mathrm{NE}}_{x-e_1,(m+1,n+1)}(t^*_{e_1} > 0)} \\
&\geq \frac{Z^{\mathrm{NE}}_{x,(m+1,n+1)} Q^{\mathrm{NE}}_{x,(m+1,n+1)}(t^*_{e_1} > 0)}{Z^{\mathrm{NE}}_{x-e_1,(m+1,n+1)}} \\
&= I_{x,(m+1,n+1)} Q^{\mathrm{NE}}_{x,(m+1,n+1)}\big(t^*_{e_1} > 0\big)
\end{aligned}
$$

and a switch to complements, we deduce that the probability on line (4.35) is bounded by

$$
\mathbb{P}\big\{Q^{\mathrm{NE}}_{x,(m+1,n+1)}\big(t^*_{e_2} > 0\big) \geq \varepsilon\big\}. \tag{4.37}
$$

This probability can be shown to be bounded by e^{-aN} for a constant $a > 0$ exactly as was done for probability (4.22), where N is defined by (4.18) and is proportional to both m and n. This time $\alpha < \lambda$, and so the characteristic direction $(\bar{m}, \bar{n})$ for (α, ρ) defined as in (4.21) satisfies $\bar{m} < m - N\gamma$ and $\bar{n} > n + N\gamma$ for some $\gamma > 0$. Qualitatively speaking this means that in (4.37) the direction (m, n) proceeds too fast along the e_1-direction, compared with the characteristic direction, and thereby renders the event $t^*_{e_2} > 0$ a deviation.

Last we need to control the moment on line (4.35). Let $x = (k, \ell) \in \mathbb{Z}_+^2$. From the definition of the ratio weights in (4.28) and the partition functions in (4.27), with superscript $[\alpha]$ dropped to simplify notation,

$$\frac{1}{\check{I}^{[\alpha]}_{x,(m,n)}} = \frac{\check{Z}_{x-e_1,(m,n)}}{\check{Z}_{x,(m,n)}} = \sum_{b=0}^{n-\ell}\biggl(\prod_{j=0}^{b} \check{\sigma}^{-1}_{x-e_1+je_2}\biggr)\frac{\check{Z}_{x+be_2,(m,n)}}{\check{Z}_{x,(m,n)}}$$

$$\le \sum_{b=0}^{n-\ell} \check{\sigma}^{-1}_{x-e_1+be_2}\prod_{j=0}^{b-1}\frac{\check{\sigma}_{x+je_2}}{\check{\sigma}_{x-e_1+je_2}} \le 2n \cdot e^{\max_{0\le b\le n} S_b} \cdot \max_{0\le b\le n} \check{\sigma}^{-1}_{x-e_1+be_2},$$

where $S_t = \sum_{j=0}^{t-1}(\log\check{\sigma}_{x+je_2} - \log\check{\sigma}_{x-e_1+je_2})$ is a sum of mean-zero i.i.d. variables with all moments. Consequently

$$\mathbb{E}\biggl|\log^+ \frac{1}{\check{I}^{[\alpha]}_{x,(m,n)}}\biggr|^{2p} \le C\log n + \mathbb{E}\Bigl[\max_{0\le b\le n} |S_b|^{2p}\Bigr] + \mathbb{E}\Bigl[\max_{0\le b\le n} |\log\check{\sigma}_{be_2}|^{2p}\Bigr]$$

$$\le Cn^{2p} \le CN^{2p}.$$

Combining the two last paragraphs shows that

$$\text{line (4.35)} \le CN^{2p}e^{-aN} \le C(p).$$

Combining all the bounds verifies (4.34) and thereby the L^p convergence in (4.4). □

5. Busemann functions and a variational characterization of the free energy. In this section we turn the limits of ratios of point-to-point partition functions into Busemann functions, and use these to solve a variational formula for the limiting free energy. The parts from this section needed for the sequel are definition (5.9) of the velocity $\mathbf{u}(h)$ associated to a tilt h, and the large deviation bound (5.15). The latter is needed for the proofs in Section 6.

We consider briefly general i.i.d. weights $w = (w_x)_{x\in\mathbb{Z}_+^2}$ on a probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ assumed to satisfy

$$\exists\varepsilon > 0: \qquad \mathbb{E}\bigl(|\log w_0|^{2+\varepsilon}\bigr) < \infty. \tag{5.1}$$

Later we specialize back to $w_0 \sim \text{Gamma}(\rho)$. It is convenient to use exponential Boltzmann–Gibbs factors. Let $p(e_1) = p(e_2) = 1/2$ be the kernel of the background random walk X_n with expectation E and initial point $X_0 = 0$. Define the potential $g(w) = -\log w_0 + \log 2$. In this notation the point-to-point partition function (4.1) is

$$Z_{0,v} = E\bigl[e^{\sum_{k=0}^{n-1} g(T_{X_k}\omega)}, X_n = v\bigr], \qquad n = |v|_1.$$

Introduce a tilted point-to-line partition function

$$Z^h_{0,(N)} = E\bigl[e^{\sum_{k=0}^{N-1} g(T_{X_k}\omega)+h\cdot X_N}\bigr], \qquad h = (h_1, h_2) \in \mathbb{R}^2 \text{ and } N \in \mathbb{N}. \tag{5.2}$$

The set of limit velocities for admissible walks in $\mathbb{Z}_+^2$ is $\mathcal{U} = \{(u, 1-u) : 0 \le u \le 1\}$, with relative interior $\operatorname{int}\mathcal{U} = \{(u, 1-u) : 0 < u < 1\}$. For each $\mathbf{u} = (u, 1-u) \in \mathcal{U}$, let $\hat{x}_n(\mathbf{u}) = (\lfloor nu \rfloor, n - \lfloor nu \rfloor)$. Define limiting point-to-point free energies

$$\Lambda_{p2p}(\mathbf{u}) = \lim_{n\to\infty} n^{-1} \log Z_{0,\hat{x}_n(\mathbf{u})}, \qquad \mathbf{u} \in \mathcal{U},$$

and tilted point-to-line free energies

$$\Lambda_{p2\ell}(h) = \lim_{N\to\infty} N^{-1} \log Z^h_{0,(N)}, \qquad h = (h_1, h_2) \in \mathbb{R}^2.$$

Under assumption (5.1) these limits exist $\mathbb{P}$-a.s., Λ_{p2p} is continuous and concave in $\mathbf{u}$ and $\Lambda_{p2\ell}$ is continuous and convex in h [25].

We recall two variational formulas, valid for i.i.d. weights under assumption (5.1). First, a convex duality between the free energies ([25], Remark 4.2, also proved below in (5.14))

$$\Lambda_{p2p}(\mathbf{u}) = \inf_{h\in\mathbb{R}^2} \{\Lambda_{p2\ell}(h) - \mathbf{u} \cdot h\}. \tag{5.3}$$

Let $\mathscr{C}_0$ denote the class of *centered cocycles* $F : \Omega \times \{e_1, e_2\} \to \mathbb{R}$ that satisfy $F \in L^1$, $\mathbb{E}F(w, z) = 0$ for $z \in \{e_1, e_2\}$, and a cocycle property $F(w, e_1) + F(T_{e_1} w, e_2) = F(w, e_2) + F(T_{e_2} w, e_1)$ $\mathbb{P}$-a.s. Then we have the variational formula ([26], Theorem 2.3),

$$\Lambda_{p2\ell}(h) = \inf_{F\in\mathscr{C}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{w} \log \sum_{z\in\{e_1,e_2\}} p(z) e^{g(w)+h\cdot z+F(w,z)}. \tag{5.4}$$

We solve (5.3) and (5.4) for the log-gamma model. The next corollary turns the limits of Theorem 4.1 into Busemann functions, and states the properties needed for the development that follows. Recall the function $\theta(\mathbf{u}) \in [0, \rho]$ of (3.13), the unique parameter such that $\mathbf{u}$ is the characteristic direction for $(\theta(\mathbf{u}), \rho)$.

COROLLARY 5.1 (Corollary of Theorem 4.1). *Assume* $\{w_x\}$ *are i.i.d.* Gamma(ρ).

(a) *For each velocity* $\mathbf{u} \in \operatorname{int}\mathcal{U}$ *and for each* $x, v \in \mathbb{Z}_+^2$, *the* $\mathbb{P}$*-almost sure limit*

$$B^{\mathbf{u}}(w, x) = \lim_{n\to\infty} (\log Z_{0,\hat{x}_n(\mathbf{u})+v} - \log Z_{x,\hat{x}_n(\mathbf{u})+v}) \tag{5.5}$$

exists and is independent of v.

(b) *The sequences* $\{B^{\mathbf{u}}(T_{ie_1} w, e_1) : i \in \mathbb{Z}_+\}$ *and* $\{B^{\mathbf{u}}(T_{je_2} w, e_2) : j \in \mathbb{Z}_+\}$ *are i.i.d. with* $e^{-B^{\mathbf{u}}(w,e_1)} \sim$ Gamma$(\theta(\mathbf{u}))$ *and* $e^{-B^{\mathbf{u}}(w,e_2)} \sim$ Gamma$(\rho - \theta(\mathbf{u}))$.

We call $B^{\mathbf{u}}$ a Busemann function, by analogy with the Busemann functions of last-passage percolation which are limits of differences $G_{0,\hat{x}_n(\mathbf{u})+v} - G_{x,\hat{x}_n(\mathbf{u})+v}$. Of course we are merely re-expressing limits (4.3) in the form

$$e^{-B^{\mathbf{u}}(T_x w, e_1)} = \lim_{n\to\infty} \frac{Z_{x+e_1,\hat{x}_n(\mathbf{u})+v}}{Z_{x,\hat{x}_n(\mathbf{u})+v}} = \eta^{\theta(\mathbf{u})}_{x+e_1}.$$

The admission of the perturbation v in (5.5) gives the cocycle property,

$$B^{\mathbf{u}}(w,x)+B^{\mathbf{u}}(T_x w,y)=B^{\mathbf{u}}(w,x+y). \tag{5.6}$$

As a function of $\mathbf{u}\in\operatorname{int}\mathcal{U}$, define the tilt vector

$$\begin{aligned} h(\mathbf{u})=\big(h_1(\mathbf{u}),h_2(\mathbf{u})\big)&=-\sum_{i=1}^{2}\mathbb{E}\big[B^{\mathbf{u}}(w,e_i)\big]e_i \\ &=\big(\Psi_0\big(\theta(\mathbf{u})\big),\Psi_0\big(\rho-\theta(\mathbf{u})\big)\big). \end{aligned} \tag{5.7}$$

Note that $h(\mathbf{u})$ is not well defined for $\mathbf{u}$ on the axes. $\theta(\mathbf{u})$ converges to 0 (to ρ) as $\mathbf{u}$ approaches the y-axis (x-axis). Then one of the coordinates of $h(\mathbf{u})$ approaches $-\infty$. The function

$$\mathbf{u}=(u,1-u)\mapsto h_1(\mathbf{u})-h_2(\mathbf{u})=\Psi_0\big(\theta(\mathbf{u})\big)-\Psi_0\big(\rho-\theta(\mathbf{u})\big) \tag{5.8}$$

is a continuous, strictly increasing function from $u\in(0,1)$ onto $(-\infty,\infty)$. An inverse function to (5.7), $\mathbb{R}^2\ni h\mapsto\mathbf{u}(h)\in\operatorname{int}\mathcal{U}$, is given by

$$\begin{aligned} &\mathbf{u}=\mathbf{u}(h)\text{ uniquely characterized by the equation} \\ &h_1-h_2=\Psi_0\big(\theta(\mathbf{u})\big)-\Psi_0\big(\rho-\theta(\mathbf{u})\big). \end{aligned} \tag{5.9}$$

Note that $\mathbf{u}(h)$ is constant when h ranges along a 45 degree diagonal. If $h=0$ there is no tilt, $\mathbf{u}(0)=(1/2,1/2)$, and $\theta(\mathbf{u}(0))=\rho/2$.

From these ingredients we solve (5.3).

THEOREM 5.2. *Let $\mathbf{u}=(u,1-u)\in\operatorname{int}\mathcal{U}$. Tilt $h(\mathbf{u})$ kills the point-to-line free energy*: $\Lambda_{p2\ell}(h(\mathbf{u}))=0\ \forall u\in\operatorname{int}\mathcal{U}$. *Furthermore, $h(\mathbf{u})$ minimizes in* (5.3) *and so*

$$\Lambda_{p2p}(\mathbf{u})=-\mathbf{u}\cdot h(\mathbf{u})=-u\Psi_0\big(\theta(\mathbf{u})\big)-(1-u)\Psi_0\big(\rho-\theta(\mathbf{u})\big). \tag{5.10}$$

Define the centered cocycle

$$F^{\mathbf{u}}(w,z)=-B^{\mathbf{u}}(w,z)-h(\mathbf{u})\cdot z,\qquad z\in\{e_1,e_2\}. \tag{5.11}$$

THEOREM 5.3. *Given $h=(h_1,h_2)\in\mathbb{R}^2$, the equation*

$$h_1(\mathbf{u})-h_2(\mathbf{u})=h_1-h_2$$

determines a unique $\mathbf{u}\in\operatorname{int}\mathcal{U}$. Then $F^{\mathbf{u}}\in\mathscr{C}_0$ is a minimizer in (5.4). *The right-hand side of* (5.4) *is constant in w, so the essential supremum can be dropped*: $\mathbb{P}$-*a.s.*,

$$\begin{aligned} \Lambda_{p2\ell}(h)=\log\sum_{z\in\{e_1,e_2\}}p(z)e^{g(w)+h\cdot z+F^{\mathbf{u}}(w,z)}&=-h_2(\mathbf{u})+h_2 \\ &=-\Psi_0\big(\rho-\theta(\mathbf{u})\big)+h_2. \end{aligned} \tag{5.12}$$

REMARK 5.4. Theorem 5.2 is the third calculation of the explicit value of $\Lambda_{p2p}(\mathbf{u})$. This result was first derived in [28] together with fluctuation bounds. The simplest proof is in [15] where the minimization of the limit of the right-hand side of (3.10) is done with convex analysis. The value (5.12) of the tilted point-to-line free energy has not been computed before.

REMARK 5.5 (Large deviations). Let us observe how the duality between tilt h and velocity $\mathbf{u}$ in (5.3) is a standard large deviation duality. The tilted quenched path measure is

$$Q^h_{0,(N)}\{x_\cdot\} = \frac{1}{Z^h_{0,(N)}} e^{\sum_{k=0}^{N-1} g(T_{x_k}\omega)+h\cdot X_N} P\{x_\cdot\}. \tag{5.13}$$

The quenched large deviation rate function for the velocity is ($\mathbb{P}$-a.s.)

$$\begin{aligned} I_h(\mathbf{v}) &= -\lim_{\delta\searrow 0} \overline{\lim_{N\to\infty}} N^{-1}\log Q^h_{0,(N)}\{|N^{-1}X_N-\mathbf{v}|\le\delta\} \\ &= \Lambda_{p2\ell}(h) - h\cdot\mathbf{v} - \Lambda_{p2p}(\mathbf{v}). \end{aligned}$$

The last equality uses the continuity of Λ_{p2p} and Lemma 2.9 in [25]. The limiting logarithmic moment generating function is

$$\Lambda_{Q,h}(a) = \lim_{N\to\infty} N^{-1}\log E^{Q^h_{0,(N)}}[e^{a\cdot X_N}] = \Lambda_{p2\ell}(h+a) - \Lambda_{p2\ell}(h) \qquad \mathbb{P}\text{-a.s.}$$

By Varadhan's theorem these are convex duals of each other:

$$I_h(\mathbf{v}) = \sup_{a\in\mathbb{R}^2}\{a\cdot\mathbf{v} - \Lambda_{Q,h}(a)\} \tag{5.14}$$

which is the same as (5.3). For the next section we need the minimizer of I_h. By (3.13), (5.10) and calculus, I_h is uniquely minimized by $\mathbf{u}(h)$ defined by (5.9). Consequently the walk converges exponentially fast: for $\delta > 0$,

$$\overline{\lim_{N\to\infty}} N^{-1}\log Q^h_{0,(N)}\{|N^{-1}X_N - \mathbf{u}(h)|\ge\delta\} < 0 \qquad \mathbb{P}\text{-a.s.} \tag{5.15}$$

Function Λ_{p2p} extends naturally to all of $\mathbb{R}^2_+$ by homogeneity: $\Lambda_{p2p}(c\mathbf{u}) = c\Lambda_{p2p}(\mathbf{u})$. Part of the duality setting is that the mean of the Busemann function gives the gradient $\nabla_{\mathbf{u}}\Lambda_{p2p}(\mathbf{u}) = -h(\mathbf{u})$.

The remainder of this section proves the theorems.

PROOF OF THEOREM 5.2. That $F^{\mathbf{u}}$ is a centered cocycle is clear by (5.6). Let

$$f^{\mathbf{u}}(w,x) = \sum_{i=0}^{m-1} F^{\mathbf{u}}(T_{x_i}w, x_{i+1}-x_i) = -B^{\mathbf{u}}(w,x) - h(\mathbf{u})\cdot x$$

be the path integral of F. The admissible path $\{x_i\}_{i=0}^m$ above satisfies $x_0 = 0$ and $x_m = x$, and the cocycle property implies that $f^{\mathbf{u}}$ depends on the path only through the endpoint x. Corollary 5.1(b) verifies exactly the sufficient condition (A.6) for (A.5), for the function $F^{\mathbf{u}}$ itself. From Theorem A.3 in the Appendix,

$$\max_{x\in\mathbb{Z}_+^d : |x|_1=n} \frac{|f^{\mathbf{u}}(w,x)|}{n} \to 0 \qquad \text{a.s.}$$

This ergodic property slips $f^{\mathbf{u}}(w, X_n)$ into the exponent in the free energy limit, and shows that tilt $h(\mathbf{u})$ kills the point-to-line free energy,

$$\begin{aligned}\Lambda_{p2\ell}\big(h(\mathbf{u})\big) &= \lim_{n\to\infty} n^{-1}\log E\big[e^{\sum_{k=0}^{n-1} g(T_{X_k}w)+h(\mathbf{u})\cdot X_n}\big]\\ &= \lim_{n\to\infty} n^{-1}\log E\big[e^{\sum_{k=0}^{n-1} g(T_{X_k}w)+h(\mathbf{u})\cdot X_n+f^{\mathbf{u}}(w,X_n)}\big]\\ &= \lim_{n\to\infty} n^{-1}\log E\big[e^{\sum_{k=0}^{n-1}(g(T_{X_k}w)+h(\mathbf{u})\cdot(X_{k+1}-X_k)+F^{\mathbf{u}}(T_{X_k}w,X_{k+1}-X_k))}\big]\\ &= 0.\end{aligned} \tag{5.16}$$

The third equality uses the definition of $f^{\mathbf{u}}$ as the path integral of $F^{\mathbf{u}}$. The last equality comes from

$$\begin{aligned}&\sum_{z\in\{e_1,e_2\}} p(z)e^{g(w)+h(\mathbf{u})\cdot z+F^{\mathbf{u}}(w,z)}\\ &\quad= \sum_{z\in\{e_1,e_2\}} p(z)e^{g(w)-B^{\mathbf{u}}(w,z)}\\ &\quad= \lim_{n\to\infty}\frac{\sum_{z\in\{e_1,e_2\}} p(z)e^{g(w)}Z_{z,\hat{x}_n(\mathbf{u})}}{Z_{0,\hat{x}_n(\mathbf{u})}} = \lim_{n\to\infty}\frac{Z_{0,\hat{x}_n(\mathbf{u})}}{Z_{0,\hat{x}_n(\mathbf{u})}} = 1.\end{aligned} \tag{5.17}$$

Fix $\mathbf{u}\in\mathcal{U}$. Since $|X_n|_1 = n$, the expression on the right-hand side of (5.3) satisfies

$$\Lambda_{p2\ell}(h) - \mathbf{u}\cdot h = \Lambda_{p2\ell}(h_1-h_2, 0) - \mathbf{u}\cdot(h_1-h_2,0)$$

and so, as a function of h, is constant along 45 degree diagonals. So the minimization needs one h point from each diagonal, which is what parameterization $h(\mathbf{v})$ of (5.7) achieves by virtue of the bijection (5.8). The upshot is that

$$\begin{aligned}\Lambda_{p2p}(\mathbf{u}) &= \inf_{\mathbf{v}\in\operatorname{int}\mathcal{U}}\big\{\Lambda_{p2\ell}\big(h(\mathbf{v})\big) - h(\mathbf{v})\cdot\mathbf{u}\big\}\\ &= \inf_{\mathbf{v}\in\operatorname{int}\mathcal{U}}\big\{-h(\mathbf{v})\cdot\mathbf{u}\big\} = -h(\mathbf{u})\cdot\mathbf{u}.\end{aligned}$$

The last step is calculus: from explicit formula (5.7), $h(\mathbf{v})\cdot\mathbf{u}$ is uniquely maximized at $\mathbf{v}=\mathbf{u}$. This completes the proof of Theorem 5.2. $\square$

PROOF OF THEOREM 5.3. Since $|X_n|_1 = n$ and by (5.16),

$$\begin{aligned}\Lambda_{p2\ell}(h) &= \lim_{n\to\infty} n^{-1}\log E\big[e^{\sum_{k=0}^{n-1} g(T_{X_k}w)+h\cdot X_n}\big]\\ &= \lim_{n\to\infty} n^{-1}\log E\big[e^{\sum_{k=0}^{n-1} g(T_{X_k}w)+h(\mathbf{u})\cdot X_n}\big] - h_2(\mathbf{u}) + h_2 = -h_2(\mathbf{u}) + h_2.\end{aligned}$$

On the other hand, by (5.17),

$$\log \sum_z p(z) e^{g(w)+h\cdot z+F^{\mathbf{u}}(w,z)} = \log \sum_z p(z) e^{g(w)+h(\mathbf{u})\cdot z+F^{\mathbf{u}}(w,z)} - h_2(\mathbf{u}) + h_2$$
$$= -h_2(\mathbf{u}) + h_2. \qquad \square$$

6. Limits of ratios of point-to-line partition functions. Armed with the limits of Theorem 4.1 and the large deviation bound of Remark 5.5, we prove convergence of ratios of tilted point-to-line partition functions. With the tilt parameter $h = (h_1, h_2) \in \mathbb{R}^2$ and $Z_{u,v}$ defined as in (4.1), let

$$Z^h_{x,(N)} = \sum_{v \in x+\mathbb{Z}^2_+ : |v|_1 = N} e^{h\cdot(v-x)} Z_{x,v} \qquad \text{for } N \in \mathbb{N} \text{ and } |x|_1 \le N.$$

This is the same as (5.2) with a general initial point x. Recall definition (5.9) that associates a velocity $\mathbf{u}(h) = (u(h), 1 - u(h))$ to a tilt h, and definition (3.13) that associates a parameter $\theta(\mathbf{v})$ to a velocity $\mathbf{v}$.

THEOREM 6.1. *Fix $0 < \rho < \infty$, and let i.i.d.* Gamma(ρ) *weights $\{w_x\}_{x\in\mathbb{Z}^2_+}$ be given. For $\lambda \in (0, \rho)$, let $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$ be the gamma system constructed in Theorem 4.1. Then for $h = (h_1, h_2) \in \mathbb{R}^2$, $x \in \mathbb{N} \times \mathbb{Z}_+$, $y \in \mathbb{Z}_+ \times \mathbb{N}$, $\mathbb{P}$-a.s.,*

$$\lim_{N\to\infty} \frac{Z^h_{x,(N)}}{e^{-h_1} Z^h_{x-e_1,(N)}} = \eta_x^{\theta(\mathbf{u}(h))} \quad \textit{and}$$
$$\lim_{N\to\infty} \frac{Z^h_{x,(N)}}{e^{-h_2} Z^h_{x-e_2,(N)}} = \zeta_x^{\theta(\mathbf{u}(h))}. \tag{6.1}$$

In other words, the limit of ratios of point-to-line partition functions tilted by h is equal to the limit of ratios of point-to-point partition functions in the direction $\mathbf{u}(h)$

$$\lim_{N\to\infty} \frac{Z^h_{x,(N)}}{e^{-h_1} Z^h_{x-e_1,(N)}} = \lim_{(m,n)\to\infty} \frac{Z_{x,(m,n)}}{Z_{x-e_1,(m,n)}}$$

and

$$\lim_{N\to\infty} \frac{Z^h_{x,(N)}}{e^{-h_2} Z^h_{x-e_2,(N)}} = \lim_{(m,n)\to\infty} \frac{Z_{x,(m,n)}}{Z_{x-e_2,(m,n)}},$$

provided $m/n \to u(h)/(1 - u(h))$. We see the duality between tilt and velocity from Remark 5.5 again. We do not presently have a proof of L^p convergence as we did for the point-to-point case in (4.4).

In Section 7 the limits of ratios from Theorems 4.1 and 6.1 give convergence of polymer measures to random walk in a correlated random environment. The remainder of this section proves Theorem 6.1.

PROOF OF THEOREM 6.1. We prove (6.1) for the horizontal ratios (first limit). Begin with a lower bound, and let $\delta_0 > 0$.

$$
\begin{aligned}
&\frac{Z^h_{x,(N)}}{e^{-h_1} Z^h_{x-e_1,(N)}} \\
&\quad = \sum_{v\,:\,|v|_1=N} \frac{e^{h\cdot(v-x)} Z_{x-e_1,v}}{e^{-h_1} Z^h_{x-e_1,(N)}} \cdot \frac{Z_{x,v}}{Z_{x-e_1,v}} \\
&\quad = \sum_{v\,:\,|v|_1=N} Q^h_{x-e_1,(N)}\{X_{N-|x|_1+1} = v\} \frac{Z_{x,v}}{Z_{x-e_1,v}} \\
&\quad \geq \sum_{m\,:\,|m-Nu(h)|<N\delta_0} Q^h_{x-e_1,(N)}\{X_{N-|x|_1+1} = (m, N-m)\} \frac{Z_{x,(m,N-m)}}{Z_{x-e_1,(m,N-m)}}.
\end{aligned}
$$

Above we introduced a tilted quenched point-to-line polymer measure

$$
Q^h_{y,(N)}\{x_\cdot\} = \frac{1}{Z^h_{y,(N)}} e^{h\cdot(x_{N-|y|_1}-y)} \prod_{i=0}^{N-|y|_1-1} w_{x_i}^{-1} \tag{6.2}
$$

for paths $x_\cdot$ from $x_0 = y$ to the line $|x_{N-|y|_1}|_1 = N$.

Apply construction (4.10) to the gamma system $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$ to define partition functions Z^λ and associated polymer measures Q^λ with northern boundary weights $\{\eta^\lambda_{i,N-m+1}\}_{1\le i\le m+1}$ and eastern boundary weights $\{\zeta^\lambda_{m+1,j}\}_{1\le j\le N-m+1}$. Recall the dual exit points (4.13)–(4.14). By an application of Lemma A.1 (to the reversed rectangle),

$$
\begin{aligned}
\frac{Z_{x,(m,N-m)}}{Z_{x-e_1,(m,N-m)}} &\geq \frac{Z^\lambda_{x,(m+1,N-m+1)}(t^*_{e_1} > 0)}{Z^\lambda_{x-e_1,(m+1,N-m+1)}(t^*_{e_1} > 0)} \\
&\geq Q^\lambda_{x,(m+1,N-m+1)}\{t^*_{e_1} > 0\} \frac{Z^\lambda_{x,(m+1,N-m+1)}}{Z^\lambda_{x-e_1,(m+1,N-m+1)}} \\
&= Q^\lambda_{x,(m+1,N-m+1)}\{t^*_{e_1} > 0\}\eta^\lambda_x.
\end{aligned}
$$

The last equality came from Lemma 4.3. Note the notational distinction: $Q^h_{y,(N)}$ is the tilted point-to-line polymer measure, while $Q^\lambda_{x,y}$ is the point-to-point polymer measure with boundary parameter λ.

We have the lower bound

$$\tag{6.3} \begin{aligned} \frac{Z^h_{x,(N)}}{e^{-h_1} Z^h_{x-e_1,(N)}} \geq \sum_{m\,:\,|m-Nu(h)|<N\delta_0} & Q^h_{x-e_1,(N)}\{X_{N-|x|_1+1} = (m, N-m)\} \\ & \times Q^\lambda_{x,(m+1,N-m+1)}\{t^*_{e_1} > 0\}\eta^\lambda_x. \end{aligned}$$

Let $0 < \lambda < \theta(\mathbf{u}(h))$. Define parameter $M \nearrow \infty$ by $N(1-u(h)) = M\Psi_1(\theta(\mathbf{u}(h)))$. Let $(\bar m, \bar n) = x + (\lfloor M\Psi_1(\rho-\lambda)\rfloor, \lfloor M\Psi_1(\lambda)\rfloor)$, a velocity essentially characteristic for (λ, ρ). As m varies in the sum on the right-hand side of (6.3), let $(m_1, n_1) = (m+1, N-m+1)$. Since Ψ_1 is strictly decreasing, if we fix $\delta_0 > 0$ small enough, there exists $\varepsilon_0 > 0$ such that, for large enough N,

$$\bar n - n_1 \geq M\Psi_1(\lambda) - M\Psi_1\big(\theta\big(\mathbf{u}(h)\big)\big) - N\delta_0 - 2 \geq M\varepsilon_0$$

and

$$m_1 - \bar m \geq Nu(h) - N\delta_0 + 1 - x - M\Psi_1(\rho - \lambda) \geq M\varepsilon_0.$$

On the second line above we also use definition (3.13) of $u(h)$.

Following the idea of Lemma 4.4 and (4.22),

$$\begin{aligned} &\mathbb{P}\big[Q^\lambda_{x,(m+1,N-m+1)}\{t^*_{e_2} > 0\} > e^{-\delta_1\varepsilon_0 M}\big] \\ &\qquad \leq \mathbb{P}\big[Q^\lambda_{x,(\bar m,\bar n)}\{t^*_{e_2} > M\varepsilon_0\} > e^{-\delta_1\varepsilon_0 M}\big] \leq e^{-c_1\varepsilon_0 M}. \end{aligned}$$

Since there are $O(N)$ m-values, Borel–Cantelli and (6.3) give, for large enough n,

$$\frac{Z^h_{x,(N)}}{e^{-h_1} Z^h_{x-e_1,(N)}} \geq \eta^\lambda_x\big(1 - e^{-\delta_1\varepsilon_0 M}\big) Q^h_{x-e_1,(N)}\big\{\big|X_{N-|x|_1+1} - N\mathbf{u}(h)\big| < N\delta_0\big\}.$$

By the quenched LDP (5.15) for the point-to-line measure, the last probability tends to 1. Thus we obtain the lower bound

$$\varliminf_{N\to\infty} \frac{Z_{x,(N)}}{e^{-h_1} Z_{x-e_1,(N)}} \geq \eta^\lambda_x \nearrow \eta^{\theta(\mathbf{u}(h))}_x \qquad \text{as we let } \lambda \nearrow \theta\big(\mathbf{u}(h)\big).$$

For the upper bound we first bound summands away from the concentration point of the quenched measure:

$$\begin{aligned} &\sum_{m\,:\,|m-Nu(h)|\geq N\delta_0} \frac{e^{h\cdot((m,N-m)-x)} Z_{x,(m,N-m)}}{e^{-h_1} Z^h_{x-e_1,(N)}} \\ &\qquad \leq w_{x-e_1} \sum_{m\,:\,|m-Nu(h)|\geq N\delta_0} \frac{e^{h\cdot((m,N-m)-x)} Z_{x,(m,N-m)}}{Z^h_{x,(N)}} \\ &\qquad \leq w_{x-e_1} Q^h_{x,(N)}\big\{\big|X_{N-|x|_1} - N\mathbf{u}(h)\big| \geq N\delta_0\big\} \longrightarrow 0. \end{aligned}$$

For the remaining fractions we develop an upper bound:

$$\begin{aligned}\frac{Z_{x,(m,N-m)}}{Z_{x-e_1,(m,N-m)}} &\le \frac{Z^\lambda_{x,(m+1,N-m+1)}(t^*_{e_2}>0)}{Z^\lambda_{x-e_1,(m+1,N-m+1)}(t^*_{e_2}>0)}\\ &\le \frac{1}{Q^\lambda_{x-e_1,(m+1,N-m+1)}\{t^*_{e_2}>0\}}\cdot\frac{Z^\lambda_{x,(m+1,N-m+1)}}{Z^\lambda_{x-e_1,(m+1,N-m+1)}}\\ &= \frac{\eta^\lambda_x}{Q^\lambda_{x-e_1,(m+1,N-m+1)}\{t^*_{e_2}>0\}}.\end{aligned}$$

Combining these,

$$\begin{aligned}\frac{Z^h_{x,(N)}}{e^{-h_1}Z^h_{x-e_1,(N)}} \le \sum_{m:|m-Nu(h)|<N\delta_0} & Q^h_{x-e_1,(N)}\{X_{N-|x|_1+1}=(m,N-m)\}\\ &\times\frac{\eta^\lambda_x}{1-Q^\lambda_{x-e_1,(m+1,N-m+1)}\{t^*_{e_1}>0\}}+o(1),\end{aligned}$$

where the $o(1)$ term tends to zero $\mathbb{P}$-a.s. Proceed as for the lower bound, this time choosing $\theta(\mathbf{u}(h))<\lambda<\rho$ to show that the Q^λ-probability above vanishes exponentially fast. This completes the proof of Theorem 6.1. $\square$

7. Limits of path measures. As in Section 4, fix $\rho\in(0,\infty)$ and assume that i.i.d. Gamma(ρ) weights $w=\{w_x: x\in\mathbb{Z}^2_+\}$ are given on a probability space $(\Omega,\mathfrak{S},\mathbb{P})$. Let $Z_{u,v}$ be the point-to-point partition function defined in (4.1), with associated quenched polymer measure

$$Q_{u,v}\{x_\cdot\}=\frac{1}{Z_{u,v}}\prod_{i=0}^{|v-u|_1-1}w^{-1}_{x_i},\qquad x_\cdot\in\Pi_{u,v}.$$

Let point-to-line polymer measures be defined as before in (5.13) or (6.2).

For $\lambda\in(0,\rho)$, let $(\xi^\lambda,\eta^\lambda,\zeta^\lambda,w)$ denote the gamma system of weights constructed in Theorem 4.1. In this environment, define RWRE transitions on $\mathbb{Z}^2_+$ by

$$\pi^{w,\lambda}(x,x+e_1)=\frac{\eta^\lambda_{x+e_1}}{\eta^\lambda_{x+e_1}+\zeta^\lambda_{x+e_2}}\quad\text{and}$$
$$\pi^{w,\lambda}(x,x+e_2)=\frac{\zeta^\lambda_{x+e_2}}{\eta^\lambda_{x+e_1}+\zeta^\lambda_{x+e_2}}.\tag{7.1}$$

Let $P^{w,\lambda}$ be the quenched path measure of the RWRE started at 0. It is characterized by the initial point and transition

$$P^{w,\lambda}(X_0=0)=1,\qquad P^{w,\lambda}(X_{k+1}=y|X_k=x)=\pi^{w,\lambda}(x,y).$$

We wrote $P^{w,\lambda}$ instead of $P^{\omega,\lambda}$ because the quenched distribution is a function of the weights w, through the limits (4.3) that appear on the right in (7.1). In other words, the probability space has not been artificially augmented with the variables that appear in definition (2.12): everything comes from the single i.i.d. collection w.

Let $Z^{\lambda}_{u,v}$ denote the partition function defined by (3.9) in gamma system $(\xi^{\lambda}, \eta^{\lambda}, \zeta^{\lambda}, w)$. Adapt the notation from (3.3) in the form

$$\tau^{\lambda}_{x,x+z} = \begin{cases} \eta^{\lambda}_{x+e_1}, & z = e_1, \\ \zeta^{\lambda}_{x+e_2}, & z = e_2. \end{cases}$$

Then we can rewrite transition (7.1) as

$$\begin{aligned} \pi^{w,\lambda}(x, x+z) &= \frac{\tau^{\lambda}_{x,x+z}}{\tau^{\lambda}_{x,x+e_1} + \tau^{\lambda}_{x,x+e_2}} \\ &= \frac{(Z^{\lambda}_{0,x+z})^{-1}}{(Z^{\lambda}_{0,x+e_1})^{-1} + (Z^{\lambda}_{0,x+e_2})^{-1}}, \qquad z \in \{e_1, e_2\}. \end{aligned}$$

In other words, this RWRE is of the competition interface type defined by (2.7) in Lemma 2.2. The next theorem shows that these walks are the limits of the polymer measures on long paths, both point-to-point and point-to-line.

THEOREM 7.1. *The following weak limits of probability measures on the path space $(\mathbb{Z}^2_+)^{\mathbb{Z}_+}$ happen for $\mathbb{P}$-a.e. w.*

(i) *Let $0 < \lambda < \rho$, and suppose $(m, n) \to \infty$ in the characteristic direction of parameters (λ, ρ) as defined in (4.2). Then $Q_{0,(m,n)}$ converges to $P^{w,\lambda}$.*

(ii) *Let $h \in \mathbb{R}^2$. Then as $N \to \infty$ the tilted point-to-line measure $Q^h_{0,(N)}$ converges to $P^{w,\theta(\mathbf{u}(h))}$.*

PROOF. Fix a finite path $x_{0,M}$ with $x_0 = 0$. Then $(m, n) \geq x_M$ for large enough (m, n), and

$$\begin{aligned} Q_{0,(m,n)}\{X_{0,M} = x_{0,M}\} &= \frac{Z_{x_M,(m,n)}}{Z_{0,(m,n)}} \prod_{i=0}^{M-1} w_{x_i}^{-1} \underset{(m,n)\to\infty}{\longrightarrow} \prod_{i=0}^{M-1} \frac{\tau^{\lambda}_{x_i,x_{i+1}}}{w_{x_i}} \\ &= \prod_{i=0}^{M-1} \pi^{w,\lambda}(x_i, x_{i+1}) = P^{w,\lambda}\{X_{0,M} = x_{0,M}\}. \end{aligned} \tag{7.2}$$

We applied limits (4.3) and used property $w_x = \eta^{\lambda}_{x+e_1} + \zeta^{\lambda}_{x+e_2}$ of the gamma system $(\xi^{\lambda}, \eta^{\lambda}, \zeta^{\lambda}, w)$ from Theorem 4.1. There are countably many finite paths and these determine weak convergence on the path space. Hence $\mathbb{P}$-a.s. limits (7.2) give claim (i).

The proof of (ii) is the same with limits (6.1) instead. □

The RWRE $P^{w,\lambda}$ has the fluctuation exponent of the $1+1$ dimensional KPZ (Kardar–Parisi–Zhang) universality class: under the averaged distribution, at time n, the typical fluctuation away from the characteristic velocity of (λ, ρ) is of size $n^{2/3}$. The reason is that the RWRE is close to a polymer, and we can apply fluctuation results for the shift-invariant log-gamma polymer. Below $\mathbb{E}$ denotes expectation over the weights w. Recall the characteristic velocity $\mathbf{u}_{\lambda,\rho}$ from (3.12).

THEOREM 7.2. *There exist constants $C_1, C_2 < \infty$ such that for $N \in \mathbb{N}$ and $b \ge C_1$,*

$$\mathbb{E}P^{w,\lambda}\{|X_N - N\mathbf{u}_{\lambda,\rho}| \ge bN^{2/3}\} \le C_2 b^{-3}. \tag{7.3}$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varlimsup_{N\to\infty} \mathbb{E}P^{w,\lambda}\{|X_N - N\mathbf{u}_{\lambda,\rho}| \le \delta N^{2/3}\} \le \varepsilon. \tag{7.4}$$

PROOF. For each N let $(m,n) = (\lfloor cN\Psi_1(\rho-\lambda)\rfloor, \lfloor cN\Psi_1(\lambda)\rfloor)$ where $c > 0$ is fixed large enough so that $m \wedge n > 2N$. Define $0 < \kappa < 1$ by $\kappa^{-1} = c(\Psi_1(\rho - \lambda) + \Psi_1(\lambda))$. Then up to errors from integer parts $(\kappa m, \kappa n) = N\mathbf{u}_{\lambda,\rho}$. (See Figure 3.)

Fix (m,n). We couple the RWRE $P^{w,\lambda}$ with the polymer that obeys the quenched distribution $Q^{\lambda\mathrm{NE}}_{0,(m,n)}$ defined by applying construction (4.10) to the gamma system $(\xi^\lambda, \eta^\lambda, \zeta^\lambda, w)$. In other words, the boundary weights η^λ and ζ^λ are on the north and east, the bulk weights come from w and the distribution of the weights is described by (3.6), with w taking on the role of $\check{\xi}$. This is the stationary log-gamma polymer to which results from [28] apply.

Define the path $\check{X}_\cdot \in \Pi_{0,(m,n)}$ by letting it follow the RWRE until it hits either the north or the east boundary of the rectangle $\{0,\dots,m\} \times \{0,\dots,n\}$, and then

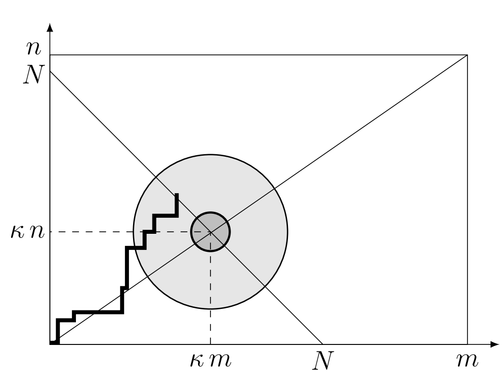

FIG. 3. *Illustration of the proof of Theorem 7.2. The thickset RWRE path avoids the disk of radius $\delta N^{2/3}$ (dark grey small disk) but enters the disk of radius $bN^{2/3}$ (light grey large disk) centered at $(\kappa m, \kappa n) = N\mathbf{u}_{\lambda,\rho}$.*

follow the boundary to (m, n). The next calculation shows that the quenched distribution of $\check{X}_\cdot$ is $Q^{\lambda\mathrm{NE}}_{0,(m,n)}$. Let $x_\cdot \in \Pi_{0,(m,n)}$. To be concrete, let $0 \le k < m$ and suppose $x_\cdot$ hits the north boundary at $x_{k+n} = (k, n)$:

$$\begin{aligned}
P^{w,\lambda}(\check{X}_\cdot = x_\cdot) &= \prod_{j=0}^{k+n-1} \frac{\tau^{\lambda}_{x_j, x_{j+1}}}{w_{x_j}} = \frac{1}{Z^{\lambda}_{0,(k,n)}} \prod_{j=0}^{k+n-1} w^{-1}_{x_j} \\
&= \frac{1}{Z^{\lambda}_{0,(m,n)}} \prod_{j=0}^{k+n-1} w^{-1}_{x_j} \cdot \prod_{i=k+1}^{m} (\eta^{\lambda}_{i,n})^{-1} \\
&= \frac{1}{Z^{\lambda\mathrm{NE}}_{0,(m,n)}} \prod_{j=0}^{k+n-1} w^{-1}_{x_j} \cdot \prod_{i=k+1}^{m} (\eta^{\lambda}_{i,n})^{-1} = Q^{\lambda\mathrm{NE}}_{0,(m,n)}\{x_\cdot\}.
\end{aligned}$$

The last equality is the definition of $Q^{\lambda\mathrm{NE}}_{0,(m,n)}\{x_\cdot\}$. The equality $Z^{\lambda\mathrm{NE}}_{0,(m,n)} = Z^{\lambda}_{0,(m,n)}$ comes by applying Lemma 4.3 to a telescoping product of ratio weights.

With c large enough, the boundary does not interfere with behavior around $(\kappa m, \kappa n) = N\mathbf{u}_{\lambda,\rho}$. In (7.3)–(7.4) we can replace $\mathbb{E}P^{w,\lambda}\{\cdot\}$ with $\mathbb{E}Q^{\lambda\mathrm{NE}}_{0,(m,n)}\{\cdot\}$. The result follows from Theorem 2.3 of [28], after a harmless reversal of the lattice rectangle to account for the difference that in ([28], Theorem 2.3), the boundary weights are on the south and west. $\square$

8. The log-gamma polymer random walk in random environment. In the previous section we saw that the limits of log-gamma polymer measures are polymer RWREs with transition (2.13), where the weights come from a gamma system with some parameters (λ, ρ). In this section we identify a stationary, ergodic probability distribution for the environment process of a polymer RWRE. We expect this stationary Markov chain to be the limit of the environment process when its initial distribution is an appropriate gamma system (Remark 8.3 below).

The process of the environment as seen from the particle is

$$T_{X_n}\omega = (\xi_{X_n+\mathbb{N}^2}, \eta_{X_n+\mathbb{N}\times\mathbb{Z}_+}, \zeta_{X_n+\mathbb{Z}_+\times\mathbb{N}}, \check{\xi}_{X_n+\mathbb{Z}^2_+}).$$

The state space of this process is the space Ω_{NE} of weight configurations $\omega = (\xi, \eta, \zeta, \check{\xi})$ that satisfy NE induction, as defined in Definition 2.3 and (2.12).

Let $0 < \alpha, \beta < \infty$ and $\rho = \alpha + \beta + 1$. Define probability distribution $\mu^{\alpha,\beta}$ on the space Ω_{NE} as follows: let the variables $(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})$ be mutually independent with marginal distributions

$$\eta_{i,0} \sim \mathrm{Gamma}(\alpha), \zeta_{0,j} \sim \mathrm{Gamma}(\beta), \xi_{i,j} \sim \mathrm{Gamma}(\rho), \qquad i, j \in \mathbb{N}. \tag{8.1}$$

The remaining variables $\{\eta_x, \zeta_x, \check{\xi}_{x-e_1-e_2} : x \in \mathbb{N}^2\}$ are then defined by north–east induction (2.8)–(2.9).

A few more notational items. G_α denotes a Gamma(α) random variable and $\mathbf{E}$ generic expectation. Let P denote the distribution of the random walk on $\mathbb{Z}_+^2$ that starts at 0 and has step distribution

$$p(e_1) = \frac{\alpha}{\alpha+\beta} = 1 - p(e_2).$$

Let us call this the $(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$ random walk. An admissible path is denoted by $x_{0,n} = (x_0, x_1, \dots, x_n)$ with $x_0 = 0$ and steps $z_k = x_k - x_{k-1} \in \{e_1, e_2\}$.

The Burke property is not valid for $\mu^{\alpha,\beta}$ because $\rho \neq \alpha + \beta$, so under $\mu^{\alpha,\beta}$ the weights do not form a gamma system (Definition 3.1). However, the $\check{\xi}$ weights still turn out to have a tractable distribution which we record in the next proposition.

PROPOSITION 8.1. *Under $\mu^{\alpha,\beta}$, the marginal distribution of $\{\check{\xi}_x\}_{x\in\mathbb{Z}_+^2}$ is given as follows. Let $\{h_x\}_{x\in\mathbb{Z}_+^2}$ be arbitrary bounded Borel functions on $\mathbb{R}_+$. Then for $n \in \mathbb{N}$,*

$$E^{\mu^{\alpha,\beta}}\Bigl[\prod_{x\in\mathbb{Z}_+^2 : |x|_1\le n} h_x(\check{\xi}_x)\Bigr]$$

$$= \sum_{x_{0,n}\in\{0\}\times(\mathbb{Z}_+^2)^n} P(X_{0,n} = x_{0,n}) \prod_{k=0}^{n} \mathbf{E}h_{x_k}(G_{\alpha+\beta}) \prod_{|y|_1\le n : y\notin\{x_{0,n}\}} \mathbf{E}h_y(G_{\alpha+\beta+1}).$$

In other words, the distribution of the $\check{\xi}$ weights is constructed as follows: run the $(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$ random walk, put independent Gamma$(\alpha+\beta)$ variables on the path, independent Gamma$(\alpha+\beta+1)$ variables off the path, and average over the walks.

THEOREM 8.2. *Let the environment ω have initial distribution $\mu^{\alpha,\beta}$ on the space Ω_{NE} of* (2.12), *and let the walk X_n obey transitions* (2.13):

(a) *The environment process $T_{X_n}\omega$ is a stationary ergodic Markov chain with state space Ω_{NE}.*

(b) *The averaged distribution of walk X_n is the homogeneous $(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$ random walk.*

Note the contrast in the behavior of the walk X_n. According to Theorem 7.2, when the environment has the distribution of a gamma system of weights, the averaged walk has fluctuations of order $n^{2/3}$. By part (b) above, when the environment has the $\mu^{\alpha,\beta}$ distribution, the averaged walk is diffusive.

REMARK 8.3 (Simulations). Suppose the environment process starts from a gamma system with parameters (λ, ρ), with $\rho > 1$. Simulations suggest that then

$T_{X_n}\omega$ converges to $\mu^{\alpha,\beta}$ such that $\alpha+\beta=\rho-1$ and $(\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta})=\mathbf{u}_{\lambda,\rho}$, the characteristic direction (3.12) of the original setting.

Under the environment distribution $\mu^{\alpha,\beta}$, the averaged distribution of the walk X_n is the diffusive $(\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta})$ random walk. Simulations suggest that under its quenched distribution the walk localizes, with a positive fraction of overlap between two independent walks in the same environment.

REMARK 8.4. We can look at the environment as seen from the walk with a more general boundary, instead of simply the axes. Let $\sigma=\{y_j\}_{j\in\mathbb{Z}}$ be a down-right path in $\mathbb{Z}^2$ that goes through e_2, 0 and e_1. That is, $y_{-1}=e_2$, $y_0=0$, $y_1=e_1$ and $y_i-y_{i-1}\in\{e_1,-e_2\}$. Let $\mathcal{J}=\{x:\exists k\in\mathbb{N}:x-(k,k)\in\sigma\}$ be the lattice strictly to the northeast of σ. Weights assigned to this setting are such that $\{\xi_x:x\in\mathcal{J}\}$ are i.i.d. Gamma(ρ). On the path edge weights have different recipes to the northwest and southeast of the origin:

horizontal edge northwest of 0: $i<0$, $y_i-y_{i-1}=e_1:\eta_{y_i}\sim\text{Gamma}(\alpha+1)$,

vertical edge northwest of 0: $i\le 0$, $y_i-y_{i-1}=-e_2:\zeta_{y_{i-1}}\sim\text{Gamma}(\beta)$,

horizontal edge southeast of 0: $i\ge 1$, $y_i-y_{i-1}=e_1:\eta_{y_i}\sim\text{Gamma}(\alpha)$,

vertical edge southeast of 0: $i>1$, $y_i-y_{i-1}=-e_2:\zeta_{y_{i-1}}\sim\text{Gamma}(\beta+1)$.

These weights are stationary as we look at the system centered at X_n. The proof goes along the same lines as given below.

REMARK 8.5 (A degenerate limit and an invariant distribution as seen from a last-passage competition interface). The results above require $\rho>1$. In the limit $\alpha\searrow 0$, $\beta\searrow 0$, $\rho\searrow 1$, the η^{-1}, ζ^{-1} weights blow up. We rescale so that logarithms of edge weights converge to exponential random variables, and bulk weights vanish. Let $\varepsilon>0$, $\rho=\varepsilon\alpha+\varepsilon\beta+1$, and consider the weights $(\xi^{(\varepsilon)}_{\mathbb{N}^2},\eta^{(\varepsilon)}_{\mathbb{N}\times\mathbb{Z}_+},\zeta^{(\varepsilon)}_{\mathbb{Z}_+\times\mathbb{N}})$ under the distribution $\mu^{\varepsilon\alpha,\varepsilon\beta}$. The independent weights of (8.1) now satisfy for $i,j\in\mathbb{N}$

$$\xi^{(\varepsilon)}_{i,j}\sim\text{Gamma}(\rho),\qquad \eta^{(\varepsilon)}_{i,0}\sim\text{Gamma}(\varepsilon\alpha),\qquad \zeta^{(\varepsilon)}_{0,j}\sim\text{Gamma}(\varepsilon\beta). \tag{8.2}$$

We can construct the weights in (8.2) as functions of uniform variables as in (4.26). Then the following limits as $\varepsilon\searrow 0$ can be taken pointwise:

$$-\varepsilon\log\xi^{(\varepsilon)}_{i,j}\to 0,\qquad -\varepsilon\log\eta^{(\varepsilon)}_{i,0}\to I_{i,0}\sim\text{Exp}(\alpha),$$

and

$$-\varepsilon\log\zeta^{(\varepsilon)}_{0,j}\to J_{0,j}\sim\text{Exp}(\beta).$$

The NE induction equations (2.8) converge to the equations

$$I_x=(I_{x-e_2}-J_{x-e_1})^+\quad\text{and}\quad J_x=(J_{x-e_1}-I_{x-e_2})^+. \tag{8.3}$$

The RWRE transition probability converges to a deterministic transition:

$$\pi^{(\varepsilon)}_{x,x+e_1} = \frac{\eta^{(\varepsilon)}_{x+e_1}}{\eta^{(\varepsilon)}_{x+e_1} + \zeta^{(\varepsilon)}_{x+e_2}} \longrightarrow \mathbf{1}\{I_{x+e_1} < J_{x+e_2}\} \equiv \pi^{(0)}_{x,x+e_1} \qquad \text{as } \varepsilon \searrow 0.$$

The limit leads to an invariant distribution for a last-passage system. Equations (8.3) describe inductively the increment variables

$$I_x = G_{0,x} - G_{0,x-e_1} \quad \text{and} \quad J_x = G_{0,x} - G_{0,x-e_2}$$

of a degenerate last-passage model with boundary weights $\{I_{i,0}, J_{0,j} : i, j \in \mathbb{N}\}$ and zero bulk weights. This distribution on $(I_{\mathbb{N}\times\mathbb{Z}_+}, J_{\mathbb{Z}_+\times\mathbb{N}})$ is invariant for the environment seen from the location φ_n that starts at $\varphi_0 = 0$ and obeys the transition

$$\pi^{(0)}_{x,x+e_1} = \mathbf{1}\{I_{x+e_1} < J_{x+e_2}\} \quad \text{and} \quad \pi^{(0)}_{x,x+e_2} = \mathbf{1}\{I_{x+e_1} > J_{x+e_2}\}. \tag{8.4}$$

Given the environment, this defines a deterministic path $\varphi_\cdot$ on $\mathbb{Z}^2_+$. We recognize in (8.4) the jump rule of the competition interface (2.2).

The remainder of this section is taken by the proofs. To prove stationarity of the Markov chain it suffices to consider the partial environment $(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})$ because the other variables of the state are functions of these. The notation here is that $\eta_{\mathbb{N}e_1} = \{\eta_{ie_1}\}_{i\in\mathbb{N}}$, and similarly for other cases. The next lemma proves everything in Proposition 8.1 and Theorem 8.2, except the ergodicity.

LEMMA 8.6. *Fix $n \in \mathbb{N}$ and an admissible path $x_{0,n}$ with $x_0 = 0$. Fix a finite set $\mathcal{I} \subset \mathbb{Z}^2_+$, disjoint from $(x_n + \mathbb{Z}^2_+) \cup \{x_k\}_{0\le k<n}$. Let $\{h_k\}_{k\in\mathbb{Z}_+}$ and $\{g_u\}_{u\in\mathbb{Z}^2_+}$ be collections of bounded Borel functions on $\mathbb{R}_+$. Let f be a bounded Borel function on $\mathbb{R}_+^{\mathbb{N}+\mathbb{N}+\mathbb{N}^2}$. Then*

$$\begin{aligned} &E^{\mu^{\alpha,\beta}}\biggl[P^\omega(X_{0,n} = x_{0,n}) \\ &\qquad \times \prod_{k=0}^{n-1} h_k(\check{\xi}_{x_k}) \cdot \prod_{u\in\mathcal{I}} g_u(\check{\xi}_u) \cdot f(\eta_{x_n+\mathbb{N}e_1}, \zeta_{x_n+\mathbb{N}e_2}, \xi_{x_n+\mathbb{N}^2})\biggr] \\ &\quad = P(X_{0,n} = x_{0,n}) \\ &\qquad \times \prod_{k=0}^{n-1} \mathbf{E}[h_k(G_{\alpha+\beta})] \cdot \prod_{u\in\mathcal{I}} \mathbf{E}[g_u(G_{\alpha+\beta+1})] \cdot E^{\mu^{\alpha,\beta}}[f(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})]. \end{aligned} \tag{8.5}$$

REMARK 8.7. Note that the independent $(\check{\xi}_{X_k})$ cannot go up to $k = n$ because $\check{\xi}_{X_n} = \eta_{X_n+e_1} + \zeta_{X_n+e_2}$, and these belong in the future of the walk. Adding the statements over $x_{0,n}$ gives the invariance of $\mu^{\alpha,\beta}$ and the distribution of $\check{\xi}$. For a fixed $x_{0,n}$ we get the averaged distribution of the walk and also the statement that when the walk looks at the $\check{\xi}$ weights in its past, it sees $G_{\alpha+\beta}$-variables on its path and $G_{\alpha+\beta+1}$-variables elsewhere.

Lemma 8.6 is basically a consequence of size-biasing beta variables. The formulation we need is in the next lemma, whose proof we leave to the reader.

LEMMA 8.8. *Let the gamma variables below with distinct subscripts be independent. Then*

$$\begin{aligned}&\mathbf{E}\biggl[\frac{G_\alpha}{G_\alpha+G_\beta}f\biggl(G_{\alpha+\beta+1}\cdot\frac{G_\alpha}{G_\alpha+G_\beta}\biggr)g\biggl(G_{\alpha+\beta+1}\cdot\frac{G_\beta}{G_\alpha+G_\beta}\biggr)h(G_\alpha+G_\beta)\biggr]\\&\quad=\frac{\alpha}{\alpha+\beta}\mathbf{E}f(G_{\alpha+1})\cdot\mathbf{E}g(G_\beta)\cdot\mathbf{E}h(G_{\alpha+\beta}).\end{aligned}\tag{8.6}$$

PROOF OF LEMMA 8.6. We assume that the first step of the walk is e_1 and calculate the distribution. Introduce functions Φ to represent north–east induction (2.8)–(2.9), specifically to calculate the $\check{\xi}$ weights on the vertical line $x\cdot e_1=0$ and ζ weights on the vertical line $x\cdot e_1=1$, for $x\cdot e_2\geq 1$,

$$\begin{aligned}(\check{\xi}_{\mathbb{N}e_2},\zeta_{e_1+\mathbb{N}e_2})&=(\check{\xi}_{\mathbb{N}e_2},\zeta_{e_1+e_2},\zeta_{e_1+e_2+\mathbb{N}e_2})\\&=\biggl(\Phi_1(\eta_{e_1+e_2},\zeta_{e_2+\mathbb{N}e_2},\xi_{e_1+e_2+\mathbb{N}e_2}),\xi_{e_1+e_2}\frac{\zeta_{e_2}}{\eta_{e_1}+\zeta_{e_2}},\\&\qquad\Phi_2(\eta_{e_1+e_2},\zeta_{e_2+\mathbb{N}e_2},\xi_{e_1+e_2+\mathbb{N}e_2})\biggr).\end{aligned}$$

Let h_0, g, f_i be bounded Borel functions of their arguments. The first equality below implements definitions. In the second equality below apply (8.6) to the triple $(G_\alpha,G_\beta,G_{\alpha+\beta+1})=(\eta_{e_1},\zeta_{e_2},\xi_{e_1+e_2})$, and note that all other variables are independent of this triple. Let $G^{\mathbb{N}e_2}_{\alpha+\beta+1}$ denote an i.i.d. Gamma$(\alpha+\beta+1)$ sequence. Augment temporarily the probability space with independent $G_{\alpha+1}$ and G_β variables that are also independent of all the other variables in f_2:

$$\begin{aligned}&E^{\mu^{\alpha,\beta}}\bigl[P^\omega(X_1=e_1)h_0(\check{\xi}_0)g(\check{\xi}_{\mathbb{N}e_2})f_1(\eta_{e_1+\mathbb{N}e_1})f_2(\zeta_{e_1+\mathbb{N}e_2})f_3(\xi_{e_1+\mathbb{N}^2})\bigr]\\&\quad=E^{\mu^{\alpha,\beta}}\biggl[\frac{\eta_{e_1}}{\eta_{e_1}+\zeta_{e_2}}h_0(\eta_{e_1}+\zeta_{e_2})f_1(\eta_{e_1+\mathbb{N}e_1})f_3(\xi_{e_1+\mathbb{N}^2})\\&\qquad\qquad\times g\biggl(\Phi_1\biggl(\xi_{e_1+e_2}\frac{\eta_{e_1}}{\eta_{e_1}+\zeta_{e_2}},\zeta_{e_2+\mathbb{N}e_2},\xi_{e_1+e_2+\mathbb{N}e_2}\biggr)\biggr)\\&\qquad\qquad\times f_2\biggl(\xi_{e_1+e_2}\frac{\zeta_{e_2}}{\eta_{e_1}+\zeta_{e_2}},\\&\qquad\qquad\qquad\Phi_2\biggl(\xi_{e_1+e_2}\frac{\eta_{e_1}}{\eta_{e_1}+\zeta_{e_2}},\zeta_{e_2+\mathbb{N}e_2},\xi_{e_1+e_2+\mathbb{N}e_2}\biggr)\biggr)\biggr]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{\alpha+\beta}\mathbf{E}\big[h_0(G_{\alpha+\beta})\big]E^{\mu^{\alpha,\beta}}\big[f_1(\eta_{e_1+\mathbb{N}e_1})\big]E^{\mu^{\alpha,\beta}}\big[f_3(\xi_{e_1+\mathbb{N}^2})\big] \\
&\quad \times E^{\mu^{\alpha,\beta}}\big[g\big(\Phi_1(G_{\alpha+1}, \zeta_{e_2+\mathbb{N}e_2}, \xi_{e_1+e_2+\mathbb{N}e_2})\big) \\
&\qquad\qquad \times f_2\big(G_\beta, \Phi_2(G_{\alpha+1}, \zeta_{e_2+\mathbb{N}e_2}, \xi_{e_1+e_2+\mathbb{N}e_2})\big)\big] \\
&= \frac{\alpha}{\alpha+\beta}\mathbf{E}\big[h_0(G_{\alpha+\beta})\big]\mathbf{E}\big[g\big(G^{\mathbb{N}e_2}_{\alpha+\beta+1}\big)\big]E^{\mu^{\alpha,\beta}}\big[f_1(\eta_{\mathbb{N}e_1})\big] \\
&\quad \times E^{\mu^{\alpha,\beta}}\big[f_2(\zeta_{\mathbb{N}e_2})\big]E^{\mu^{\alpha,\beta}}\big[f_3(\xi_{\mathbb{N}^2})\big] \\
&= \frac{\alpha}{\alpha+\beta}\mathbf{E}\big[h_0(G_{\alpha+\beta})\big]\mathbf{E}\big[g\big(G^{\mathbb{N}e_2}_{\alpha+\beta+1}\big)\big]E^{\mu^{\alpha,\beta}}\big[f_1(\eta_{\mathbb{N}e_1})f_2(\zeta_{\mathbb{N}e_2})f_3(\xi_{\mathbb{N}^2})\big].
\end{aligned}
$$

In the second-to-last equality, inside f_1 and f_3 we simply shift by $-e_1$. Inside f_2 variable G_β furnishes ζ_{e_2}. Here is the key point: at this stage the Burke property applies to the mappings (Φ_1, Φ_2) because $G_{\alpha+1}$ furnishes $\eta_{e_1+e_2}$, and thereby the parameters of the input weights satisfy $(\alpha+1)+\beta=\rho$. The beta size-biasing put us back into the setting of a gamma system. Thus (Φ_1, Φ_2) outputs two independent sequences. The first one denoted by $G^{\mathbb{N}e_2}_{\alpha+\beta+1}$ is i.i.d. Gamma$(\alpha+\beta+1)$ and it represents the distribution of $\check{\xi}_{\mathbb{N}e_2}$. The second one is i.i.d. Gamma(β), which we take to be $\zeta_{e_2+\mathbb{N}e_2}$. In the last equality we can combine the three $\mu^{\alpha,\beta}$-expectations because the independence is in accordance with the definition of $\mu^{\alpha,\beta}$.

Standard arguments generalize the product $f_1f_2f_3$ so that

$$
\begin{aligned}
&E^{\mu^{\alpha,\beta}}\big[P^\omega(X_1=e_1)h_0(\check{\xi}_0)g(\check{\xi}_{\mathbb{N}e_2})F(\eta_{e_1+\mathbb{N}e_1}, \zeta_{e_1+\mathbb{N}e_2}, \xi_{e_1+\mathbb{N}^2})\big] \\
&\qquad = p(e_1)\mathbf{E}\big[h_0(G_{\alpha+\beta})\big]\mathbf{E}\big[g\big(G^{\mathbb{N}e_2}_{\alpha+\beta+1}\big)\big]E^{\mu^{\alpha,\beta}}\big[F(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})\big]
\end{aligned}
$$

for Borel functions h_0, g, F such that the expectations make sense. Reflection across the diagonal gives the alternative formula where the first step is e_2 instead of e_1, $\check{\xi}_{\mathbb{N}e_2}$ is replaced by $\check{\xi}_{\mathbb{N}e_1}$ and $G^{\mathbb{N}e_2}_{\alpha+\beta+1}$ is replaced by $G^{\mathbb{N}e_1}_{\alpha+\beta+1}$.

Referring to the goal (8.5), let $\mathcal{I}_0 = \mathcal{I}\setminus(x_1+\mathbb{Z}^2_+)$ and take $g(\check{\xi}_\cdot) = \prod_{u\in\mathcal{I}_0} g_u(\check{\xi}_u)$. We can combine the e_1 and e_2 cases into this statement, which is (8.5) for $n=1$:

$$
\begin{aligned}
&E^{\mu^{\alpha,\beta}}\Big[P^\omega(X_1=x_1)h_0(\check{\xi}_0)\cdot\prod_{u\in\mathcal{I}_0} g_u(\check{\xi}_u)\cdot F(\eta_{x_1+\mathbb{N}e_1}, \zeta_{x_1+\mathbb{N}e_2}, \xi_{x_1+\mathbb{N}^2})\Big] \\
&\quad = p(x_1)\mathbf{E}\big[h_0(G_{\alpha+\beta})\big]\cdot\prod_{u\in\mathcal{I}_0}\mathbf{E}\big[g_u(G_{\alpha+\beta+1})\big]\cdot E^{\mu^{\alpha,\beta}}\big[F(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})\big].
\end{aligned} \tag{8.7}
$$

To obtain (8.5), do induction on the length n of the path. Let $\mathcal{I}' = \mathcal{I}\cap(x_1+\mathbb{Z}^2_+)$. In (8.7) take

$$
\begin{aligned}
F(\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2}) &= \prod_{i=1}^{n-1}\pi_{x_i-x_1, x_{i+1}-x_1}(\omega)\cdot\prod_{k=1}^{n-1} h_k(\check{\xi}_{x_k-x_1}) \\
&\quad \times \prod_{u\in\mathcal{I}'-x_1} g_{u+x_1}(\check{\xi}_u)\cdot f(\eta_{x_n-x_1+\mathbb{N}e_1}, \zeta_{x_n-x_1+\mathbb{N}e_2}, \xi_{x_n-x_1+\mathbb{N}^2}).
\end{aligned}
$$

Assuming (8.5) holds for paths of length $n-1$, the right-hand side of (8.7) turns into the right-hand side of (8.5). $\square$

The ergodicity claim of Theorem 8.2 is in the next lemma.

LEMMA 8.9. *With initial distribution $\mu^{\alpha,\beta}$, the stationary process $S_n = (\eta_{X_n+\mathbb{N}e_1}, \zeta_{X_n+\mathbb{N}e_2}, \xi_{X_n+\mathbb{N}^2})$ is ergodic.*

PROOF. Denote a generic state by $S = (\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_{\mathbb{N}^2})$. It suffices to show that, for any function $f \in L^1(\mu^{\alpha,\beta})$, the averages

$$n^{-1}\sum_{k=0}^{n-1} E^S[f(S_k)]$$

converge to a constant in $L^1(\mu^{\alpha,\beta})$ ([27], pages 91–95). By approximation in $L^1(\mu^{\alpha,\beta})$, it suffices to prove this for a local function f, that is, a function of the variables $\mathbf{s} = (\eta_{i,0}, \zeta_{0,j}, \xi_{i,j})_{i,j\in[M]}$ for an arbitrary but fixed $M \in \mathbb{N}$. Let $\mathbf{s} = \varphi(S)$ denote the projection mapping, and let the projection of the stationary process S_n be $\mathbf{s}_n = \varphi(S_n) = (\eta_{X_n+(i,0)}, \zeta_{X_n+(0,j)}, \xi_{X_n+(i,j)})_{i,j\in[M]}$.

Process $\mathbf{s}_n$ is also a stationary Markov chain, with state space $\mathbb{R}_+^{2M+M^2}$ and invariant distribution $\nu = \mu^{\alpha,\beta} \circ \varphi^{-1}$. Under ν coordinates of $\mathbf{s}$ are independent with distributions $\eta_{i,0} \sim \text{Gamma}(\alpha)$, $\zeta_{0,j} \sim \text{Gamma}(\beta)$ and $\xi_{i,j} \sim \text{Gamma}(\rho)$.

Given state $\mathbf{s} = (\eta_{i,0}, \zeta_{0,j}, \xi_{i,j})_{i,j\in[M]}$, we compute the variables $\{\eta_x, \zeta_x : x \in [M]^2\}$ via north–east induction (2.8). The transition from state $\mathbf{s}$ to a new state goes by two steps: (i) randomly shift $\mathbf{s}$ by e_1 or e_2; (ii) add fresh variables to the north or east to replace the variables lost from south or west in the shift of the $M \times M$ square.

Precisely speaking, from $\mathbf{s} = (\eta_{i,0}, \zeta_{0,j}, \xi_{i,j})_{i,j\in[M]}$ the process jumps to either $\mathbf{t}'$ or $\mathbf{t}''$, according to the following two cases:

(a) The shift is e_1 and $\mathbf{t}' = (\eta_{i+1,0}, \zeta_{1,j}, \xi_{i+1,j})_{i,j\in[M]}$ where the new independently chosen variables are $\eta_{M+1,0} \sim \text{Gamma}(\alpha)$ and $\xi_{M+1,j} \sim \text{Gamma}(\rho)$ for $j \in [M]$.

(b) The shift is e_2 and $\mathbf{t}'' = (\eta_{i,1}, \zeta_{0,j+1}, \xi_{i,j+1})_{i,j\in[M]}$ where the new independently chosen variables are $\zeta_{0,M+1} \sim \text{Gamma}(\beta)$ and $\xi_{i,M+1} \sim \text{Gamma}(\rho)$ for $i \in [M]$.

The probabilities of the two alternatives are

$$\pi(\mathbf{s}, \mathbf{t}') = \frac{\eta_{e_1}}{\eta_{e_1} + \zeta_{e_2}} \quad \text{and} \quad \pi(\mathbf{s}, \mathbf{t}'') = \frac{\zeta_{e_2}}{\eta_{e_1} + \zeta_{e_2}}.$$

Let $\pi(\mathbf{s}, d\mathbf{t})$ denote the transition probability of the Markov chain $\mathbf{s}_n$: the shift followed by the random choice of new coordinates to complete the square $[M] \times [M]$. The task is to check that $\mathbf{s}_n$ is an ergodic process.

Two general observations about checking the ergodicity of a Markov transition P with invariant distribution ν. (i) Suppose ν has a density with respect to a background measure λ. Then it is enough to check that, for ν-a.e. x, $P(x, dy)$ has a density $p(x, y)$ with respect to $\lambda(dy)$ such that $p(x, y) > 0$ for λ-a.e. y. For then, if A is a ν-a.s. invariant measurable set such that $\nu(A^c) > 0$, taking $x \in A^c$ in

$$\mathbf{1}_A(x) = P(x, A) = \int_A p(x, y)\lambda(dy)$$

shows that $\lambda(A) = 0$ and thereby $\nu(A) = 0$. (ii) It is enough to check the ergodicity of some power P^m.

We show that for $m = 2M + 1$, $\pi^m(\mathbf{s}, d\mathbf{t})$ has a Lebesgue almost everywhere positive density on $\mathbb{R}_+^{2M+M^2}$. Let B be a Borel subset of $\mathbb{R}_+^{2M+M^2}$. Write

$$T_x\mathbf{s} = \big(\eta_{x+(i,0)}, \zeta_{x+(0,j)}, \xi_{x+(i,j)} : i, j \in [M]\big) \tag{8.8}$$

for the shifted configuration in the $M \times M$ square:

$$\begin{aligned} \pi^m(\mathbf{s}, B) &= \sum_{x\in\mathbb{Z}_+^2 : |x|_1=m} E^{\mu^{\alpha,\beta}}\big[\mathbf{1}_B(T_x\mathbf{s}) P_0^\omega\{X_m = x\} \big| \varphi(S_0) = \mathbf{s}\big] \\ &= \sum_{x\in\mathbb{Z}_+^2 : |x|_1=m} E^{\mu^{\alpha,\beta}}\big[E^{\mu^{\alpha,\beta}}\{\mathbf{1}_B(T_x\mathbf{s})|\mathcal{H}_x\} P_0^\omega\{X_m = x\} \big| \varphi(S_0) = \mathbf{s}\big]. \end{aligned} \tag{8.9}$$

On the first line above, $E^{\mu^{\alpha,\beta}}$ represents the choices of fresh coordinates while the shifts are in the quenched probability $P_0^\omega\{X_m = x\}$. After that we conditioned on the σ-algebra (Figure 4)

$$\mathcal{H}_x = \sigma\big\{\eta_{\mathbb{N}e_1}, \zeta_{\mathbb{N}e_2}, \xi_x, \{\xi_{i,j} : i \le x \cdot e_1 - 1 \text{ or } j \le x \cdot e_2 - 1\}\big\}. \tag{8.10}$$

$X_m = x$ implies $|x|_1 = m$, and then $m = 2M + 1$ guarantees that $\mathcal{H}_x$ is large enough to contain the event $\varphi(S_0) = \mathbf{s}$. The quenched probability $P_0^\omega\{X_m = x\}$

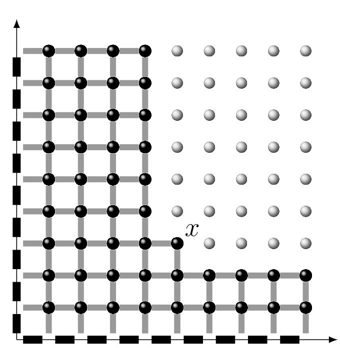

FIG. 4. *The σ-algebra $\mathcal{H}_x$. The dark black sites in the interior and the thickset lines on the axes denote the $\{\xi, \eta, \zeta\}$ variables that generate $\mathcal{H}_x$. The gray lines denote $\{\eta, \zeta\}$ variables computed via north–east induction from information contained in $\mathcal{H}_x$. Finally the lighter gray sites denote ξ variables independent of $\mathcal{H}_x$.*

is also $\mathcal{H}_x$-measurable. Of the variables that make up $T_x\mathbf{s}$ in (8.8), the $\xi_{x+(i,j)}$'s are independent of $\mathcal{H}_x$, but the $\eta_{x+(i,0)}$'s and $\zeta_{x+(0,j)}$'s depend on $\mathcal{H}_x$ through the equations

$$\eta_{x+(i,0)} = \xi_{x+(i,0)} \frac{\eta_{x+(i,-1)}}{\eta_{x+(i,-1)} + \zeta_{x+(i-1,0)}}, \qquad i = 1, \dots, M \tag{8.11}$$

and

$$\zeta_{x+(0,j)} = \xi_{x+(0,j)} \frac{\zeta_{x+(-1,j)}}{\eta_{x+(0,j-1)} + \zeta_{x+(-1,j)}}, \qquad j = 1, \dots, M.$$

The situations for $\{\eta_{x+(i,0)}\}$ and $\{\zeta_{x+(0,j)}\}$ are symmetric, so let us look at equation (8.11) closely. $\mathcal{H}_x$ contains variables $\{\zeta_x; \eta_{x+(i,-1)} : i \in [M]\}$ because these can be computed by north–east induction from the variables listed in (8.10), so these are taken as given in (8.11). Variables $\{\xi_{x+(i,0)} : i \in [M]\}$ are picked i.i.d. Gamma(ρ), independently of $\mathcal{H}_x$, while variables $\{\zeta_{x+(i-1,0)} : i = 2, \dots, M\}$ are calculated along the way from the equations

$$\zeta_{x+(i-1,0)} = \xi_{x+(i-1,0)} \frac{\zeta_{x+(i-2,0)}}{\eta_{x+(i-1,-1)} + \zeta_{x+(i-2,0)}}, \qquad i = 2, \dots, M. \tag{8.12}$$

Regarding $\{\zeta_x; \eta_{x+(i,-1)} : i \in [M]\}$ as given parameters, equations (8.11) and (8.12) show that the vectors $\bar{\eta} = (\eta_{x+(i,0)} : i \in [M])$ and $\bar{\xi} = (\xi_{x+(i,0)} : i \in [M])$ in $(0, \infty)^M$ are bijective functions of each other, and these functions are rational functions with positive coefficients. (The coefficients themselves are functions of $\{\zeta_x; \eta_{x+(i,-1)} : i \in [M]\}$.) Thus the Jacobians of these functions cannot vanish on $(0, \infty)^M$. Consequently, from the everywhere positive density of $\bar{\xi}$ [product of Gamma(ρ) distributions], we get an everywhere positive density f_1 for $\bar{\eta}$, for every given value of $\{\zeta_x; \eta_{x+(i,-1)} : i \in [M]\}$.

This argument can be repeated to get an everywhere positive density f_2 for the vector $\bar{\zeta} = (\zeta_{x+(0,j)} : i \in [M])$, for every given value of the variables specified by the conditioning on $\mathcal{H}_x$.

Let f denote the (everywhere positive) density of the vector $(\xi_{x+(i,j)} : i, j \in [M])$. With this notation we can write

$$E^{\mu^{\alpha,\beta}}[\mathbf{1}_B(T_x\mathbf{s}) | \mathcal{H}_x] = \int_{\mathbb{R}_+^{2M+M^2}} \mathbf{1}_B(u, v, w) f_1(u) f_2(v) f(w)\, du\, dv\, dw,$$

where the right-hand side is not a constant, but the densities f_1 and f_2 depend also on the variables specified by the conditioning on $\mathcal{H}_x$. On the right the densities are multiplied due to independence that comes from dependence on disjoint sets of ξ variables. This formula can be substituted into (8.9) to conclude that $\pi^m(\mathbf{s}, \cdot)$ has an a.e. positive density on $(0, \infty)^{2M+M^2}$. $\square$

APPENDIX: AUXILIARY RESULTS

This appendix contains a comparison lemma for partition functions, a large deviation bound for the log-gamma polymer and an ergodic theorem for cocycles.

A.1. Comparison lemma for partition functions. Let arbitrary weights $\{V_x\}_{x\in\mathbb{Z}_+^2}$ be given, and define partition functions as in (2.3). For a subset $A \subseteq \Pi_{u,v}$, define the restricted partition function (unnormalized polymer measure) by

$$Z_{u,v}(A) = \sum_{x.\in A} \prod_{i=1}^{|v-u|_1} V_{x_i}^{-1}.$$

Recall the definitions of the exit points (3.7)–(3.8). The restriction $A = \{t_{e_1} > 0\}$ means that the first step of the path is e_1. In other words, $Z_{0,x}(t_{e_1} > 0) = V_{e_1}^{-1} Z_{e_1,x}$, defined for $x \cdot e_1 \geq 1$.

LEMMA A.1. *For $m \geq 2$ and $n \geq 1$ we have this comparison of partition functions*:

$$\frac{Z_{0,(m-1,n)}(t_{e_1} > 0)}{Z_{0,(m,n)}(t_{e_1} > 0)} \leq \frac{Z_{(1,1),(m-1,n)}}{Z_{(1,1),(m,n)}} \leq \frac{Z_{0,(m-1,n)}(t_{e_2} > 0)}{Z_{0,(m,n)}(t_{e_2} > 0)}. \tag{A.1}$$

PROOF. Consider the ratio weights for these partition functions:

$$\eta_x = \frac{Z_{0,x-e_1}(t_{e_1} > 0)}{Z_{0,x}(t_{e_1} > 0)} = \frac{Z_{e_1,x-e_1}}{Z_{e_1,x}} \quad \text{and} \quad \tilde{\eta}_x = \frac{Z_{(1,1),x-e_1}}{Z_{(1,1),x}},$$
$$\zeta_x = \frac{Z_{0,x-e_2}(t_{e_1} > 0)}{Z_{0,x}(t_{e_1} > 0)} = \frac{Z_{e_1,x-e_2}}{Z_{e_1,x}} \quad \text{and} \quad \tilde{\zeta}_x = \frac{Z_{(1,1),x-e_2}}{Z_{(1,1),x}}.$$

On the boundary of the lattice $\mathbb{N}^2$, these ratios satisfy

$$\zeta_{1,j} = V_{1,j} = \tilde{\zeta}_{1,j} \quad \text{and} \quad \eta_{i,1} = V_{i,1}\frac{\eta_{i,0}}{\eta_{i,0} + \zeta_{i-1,1}} < V_{i,1} = \tilde{\eta}_{i,1} \qquad \text{for } i, j \geq 2.$$

NE induction (2.8) preserves these inequalities and gives the first inequality of (A.1). The second comes analogously. $\square$

A.2. Large deviation bound for the log-gamma polymer. Let $0 < \alpha < \rho$, and let (ξ, η, ζ) be a gamma system of weights with parameters (α, ρ) according to Definition 3.1. Let $Z_{0,v}$ be the partition function defined by (3.9) in this gamma system, with the corresponding point-to-point quenched polymer measure

$$Q_{0,v}\{x.\} = \frac{1}{Z_{0,v}}\left(\prod_{i=1}^{t_{\text{exit}}} \tau_{\{x_{i-1},x_i\}}^{-1}\right)\left(\prod_{j=t_{\text{exit}}+1}^{|v|_1} \xi_{x_j}^{-1}\right), \qquad x. \in \Pi_{0,v}.$$

Let the scaling parameter $N \geq 1$ be real valued. Let $(m,n) \in \mathbb{N}^2$ denote the endpoint of the path. Measure the deviation from characteristic velocity by

$$\kappa_N = \left|m - N\Psi_1(\rho - \alpha)\right| \vee \left|n - N\Psi_1(\alpha)\right|. \tag{A.2}$$

LEMMA A.2. *Let κ_N be defined by* (A.2). *Let $\delta > 0$. Then there are constants $0 < \delta_1, c, c_1 < \infty$ such that the following estimate holds. For $(m,n) \in \mathbb{N}^2$, $N \geq 1$ and $u \geq (1 \vee c\kappa_N \vee \delta N)$,*

$$\mathbb{P}\big[Q_{0,(m,n)}\{t_{e_1} \geq u\} \geq e^{-\delta_1 u}\big] \leq e^{-c_1 u}.$$

Same bound holds for t_{e_2}. The same constants work for (α, ρ) that satisfy $0 < \alpha < \rho$ and vary in a compact set.

PROOF. Let $\beta < \alpha$, and take two gamma systems: $(\xi, \eta^\alpha, \zeta^\alpha)$ with parameters (α, ρ) and $(\xi, \eta^\beta, \zeta^\beta)$ with parameters (β, ρ). Couple them so that they share the ξ-variables, and $\eta^\beta_x \leq \eta^\alpha_x$ and $\zeta^\beta_x \geq \zeta^\alpha_x$ hold. This can be achieved by imposing these same conditions on the variables in part (c) of Definition 3.1, and then noting that the inequalities are preserved by (3.1). Let Z^α and Z^β be partition functions computed in these two systems:

$$\begin{aligned}
&Q_{0,(m,n)}\{t_{e_1} \geq u\} \\
&\quad = \frac{1}{Z^\alpha_{0,(m,n)}} \sum_{x_\cdot \in \Pi_{0,(m,n)}} \mathbf{1}\{t_{e_1} \geq u\} \Bigg(\prod_{i=1}^{t_{\mathrm{exit}}} \frac{1}{\eta^\alpha_{i,0}}\Bigg)\Bigg(\prod_{j=t_{\mathrm{exit}}+1}^{m+n} \xi^{-1}_{x_j}\Bigg) \qquad \text{(A.3)}\\
&\quad \leq \frac{Z^\beta_{0,(m,n)}}{Z^\alpha_{0,(m,n)}} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{\eta^\beta_{i,0}}{\eta^\alpha_{i,0}}.
\end{aligned}$$

In the bounds below, $\overline{X} = X - \mathbb{E}X$ denotes a centered random variable. Recall the mean (3.10). Let $\delta_1 > 0$. From (A.3)

$$\begin{aligned}
&\mathbb{P}\big[Q_{0,(m,n)}\{t_{e_1} \geq u\} \geq e^{-\delta_1 u}\big] \\
&\quad \leq \mathbb{P}\Bigg\{\sum_{i=1}^{\lfloor u \rfloor}\big(\overline{\log \eta^\beta_{i,0}} - \overline{\log \eta^\alpha_{i,0}}\big) \geq \delta_1 u\Bigg\} \qquad \text{(A.4)}\\
&\qquad + \mathbb{P}\big\{\overline{\log Z^\beta_{0,(m,n)}} - \overline{\log Z^\alpha_{0,(m,n)}} \geq \big(\lfloor u \rfloor - m\big)\big(\Psi_0(\alpha) - \Psi_0(\beta)\big) \\
&\qquad\qquad + n\big(\Psi_0(\rho-\beta) - \Psi_0(\rho-\alpha)\big) - 2\delta_1 u\big\}.
\end{aligned}$$

Standard large deviations apply to log-gamma variables, so $\exists c_2 > 0$ such that

$$\mathbb{P}\Bigg\{\sum_{i=1}^{\lfloor u \rfloor}\big(\overline{\log \eta^\beta_{i,0}} - \overline{\log \eta^\alpha_{i,0}}\big) \geq \delta_1 u\Bigg\} \leq e^{-c_2 u}.$$

Taylor expand to second order the Ψ_0-differences inside the last probability in (A.4). Keeping $\delta > 0$ fixed, pick $\delta_1 > 0$ and $\alpha - \beta > 0$ small enough and $c < \infty$ large enough. Then for another small constant $c_3 > 0$, the probability simplifies to

$$\mathbb{P}\big\{\overline{\log Z^\beta_{0,(m,n)}} - \overline{\log Z^\alpha_{0,(m,n)}} \geq c_3 u\big\} \leq e^{-c_4 u}.$$

The bound comes again from i.i.d. large deviations, by virtue of (3.11). $\square$

A.3. Ergodic theorem for centered cocycles. With a bit of extra effort and with future use in mind, we prove this ergodic theorem more generally than required for this paper. Fix a dimension $d \in \mathbb{N}$. Let $\mathcal{R} \subset \mathbb{Z}^d$ denote an arbitrary finite set of admissible steps that contains at least one nonzero point. $0 \in \mathcal{R}$ is also acceptable. Admissible paths $(x_k)_{k=0}^n$ satisfy $x_k - x_{k-1} \in \mathcal{R}$. Let $M = |\mathcal{R}|$ be the cardinality of $\mathcal{R}$.

Define

$$\mathcal{G}^+ = \left\{ \sum_{z \in \mathcal{R}} b_z z : b_z \in \mathbb{Z}_+ \right\},$$

and let $\mathcal{G} = \mathcal{G}^+ - \mathcal{G}^+$ be the additive subgroup of $\mathbb{Z}^d$ generated by $\mathcal{R}$. Let $(\Omega, \mathfrak{S}, \mathbb{P})$ be a probability space equipped with a semigroup $(T_x)_{x \in \mathcal{G}^+}$ of commuting measurable maps $T_x : \Omega \to \Omega$. In other words, the assumptions are that $T_0 = \mathrm{id}$ and $T_{x+y} = T_x \circ T_y$ for $x, y \in \mathcal{G}^+$. Generic points of Ω are denoted by ω. Assume $\mathbb{P}$ invariant and ergodic under $(T_x)_{x \in \mathcal{G}^+}$: that is, $\mathbb{P} \circ T_x^{-1} = \mathbb{P}$, and if $T_x^{-1} A = A$ $\forall x \in \mathcal{G}^+$ then $\mathbb{P}(A) \in \{0, 1\}$.

Let $F : \Omega \times \mathcal{R} \to \mathbb{R}$ be a centered cocycle, by which we mean these properties:

(i) $\forall z \in \mathcal{R} : F(\omega, z) \in L^1(\mathbb{P})$ and $\mathbb{E} F(\omega, z) = 0$.

(ii) The closed-loop (or cocyle) property: if $\{x_k\}_{k=0}^n$ and $\{x'_\ell\}_{\ell=0}^m$ are two admissible paths such that $x_0 = x'_0$ and $x_n = x'_m$, then

$$\sum_{k=0}^{n-1} F(T_{x_k}\omega, x_{k+1} - x_k) = \sum_{\ell=0}^{m-1} F\big(T_{x'_\ell}\omega, x'_{\ell+1} - x'_\ell\big).$$

Note that the closed-loop property forces $F(\omega, 0) = 0$ if $0 \in \mathcal{R}$.

Define the path integral of F for $(\omega, x) \in \Omega \times \mathcal{G}$ by

$$f(\omega, x) = \sum_{k=0}^{n-1} F(T_{k_i}\omega, x_{k+1} - x_k) - \sum_{\ell=0}^{m-1} F\big(T_{x'_\ell}\omega, x'_{\ell+1} - x'_\ell\big),$$

where $(x_k)_{k=0}^n$ and $(x'_\ell)_{\ell=0}^m$ are any two admissible paths from a common initial point $x_0 = x'_0$ to $x_n = x$ and to $x'_m = 0$. In particular $f(\omega, 0) = 0$. The closed-loop property ensures that f is well defined.

Let $D_n = \{x : \exists z_1, \dots, z_n \in \mathcal{R}$ such that $z_1 + \cdots + z_n = x\}$ denote the set of points accessible from 0 in exactly n steps.

THEOREM A.3. *Let F be a centered cocycle. Assume there exists a function $\overline{F} : \Omega \times \mathcal{R} \to \mathbb{R}$ such that $F(\omega, z) \le \overline{F}(\omega, z)$ for all $z \in \mathcal{R}$ and $\mathbb{P}$-almost every ω, and that satisfies*

$$\overline{\lim_{\delta \searrow 0}}\ \overline{\lim_{n \to \infty}} \max_{|x|_1 \le n} \frac{1}{n} \sum_{0 \le i \le n\delta} \big|\overline{F}(T_{x+iz}\omega, z)\big| = 0 \qquad \forall z \in \mathcal{R} \setminus \{0\}. \tag{A.5}$$

Then for $\mathbb{P}$-almost every ω

$$\lim_{n\to\infty} \max_{x\in D_n} \frac{|f(\omega,x)|}{n} = 0.$$

An assumption similar to (A.5) was useful in [25, 26] in studies of polymers. If, for each $z \in \mathcal{R} \setminus \{0\}$, the variables $\{\overline{F}(T_{iz}\omega, z)\}_{i\in\mathbb{Z}_+}$ are i.i.d. then by Lemma A.4 of [26] a sufficient condition for (A.5) is

$$\exists p > d: \qquad \mathbb{E}\big[\big|\overline{F}(\omega, z)\big|^p\big] < \infty. \tag{A.6}$$

Our application of Theorem A.3 is to the centered cocycle $F^{\mathbf{u}}$ in (5.11). By Corollary 5.1 this satisfies the i.i.d. condition and even has an exponential moment. Thus hypothesis (A.5) is satisfied by $F^{\mathbf{u}}$ in (5.11), and Theorem A.3 holds for $F = \overline{F} = F^{\mathbf{u}}$.

As an auxiliary result toward the main theorem, we prove a limit for averages over rectangles of any dimension. The following result is a discrete version of Lemma 6.1 of [18].

THEOREM A.4. *Let F be a centered cocycle. Let $r \in [M] = \{1, 2, \dots, M\}$, $z_1, \dots, z_r$ distinct points from $\mathcal{R}$, and $0 \le a_i < b_i$ for $1 \le i \le r$. Then*

$$\lim_{n\to\infty} \frac{1}{n^r} \sum_{k_1=\lfloor na_1\rfloor}^{\lfloor nb_1\rfloor-1} \cdots \sum_{k_r=\lfloor na_r\rfloor}^{\lfloor nb_r\rfloor-1} \frac{f(\omega, k_1z_1 + \cdots + k_rz_r)}{n} = 0 \qquad \mathbb{P}\text{-a.s.} \tag{A.7}$$

It is enough to consider the case $a_i = 0$, for the general case is obtained by successive differences and sums of such cases. Then to simplify notation we take $b_i = 1$. We separate a part of the proof as a lemma.

LEMMA A.5. *Let $1 \le j \le r \le M$ and $g : [0,1]^r \to \mathbb{R}$ continuous. Then $\mathbb{P}$-a.s.*

$$\lim_{n\to\infty} \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} g\big(n^{-1}(k_1, \dots, k_r)\big) F(T_{k_1z_1+\cdots+k_rz_r}\omega, z_j) = 0. \tag{A.8}$$

PROOF. Fix j. Let $h(\omega)$ denote the a.s. limit of the left-hand side of (A.8) for $g \equiv 1$, given by the pointwise ergodic theorem ([19], Theorem 6.2.8). We show that h is invariant under each shift T_z, $z \in \mathcal{R}$. By the closed-loop property (now for $j \in \{1, \dots, r\}$)

$$\begin{aligned} &\frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} F(T_{k_1z_1+k_2z_2+\cdots+k_rz_r}\omega, z_j) \\ &\quad + \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{j-1}=0}^{n-1} \sum_{k_{j+1}=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} F(T_{k_1z_1+k_2z_2+\cdots+nz_j+\cdots+k_rz_r}\omega, z) \end{aligned}$$

$$= \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{j-1}=0}^{n-1} \sum_{k_{j+1}=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} F(T_{k_1 z_1 + k_2 z_2 + \cdots + 0 \cdot z_j + \cdots + k_r z_r} \omega, z)$$

$$+ \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} F\bigl(T_{k_1 z_1 + k_2 z_2 + \cdots + k_r z_r}(T_z \omega), z_j\bigr).$$

The closed loop above is taken for fixed $T_{k_1 z_1 + \cdots + k_{j-1} z_{j-1} + k_{j+1} z_{j+1} + \cdots + k_r z_r} \omega$. The two paths are $\{z_j, 2z_j, \ldots, nz_j, nz_j + z\}$ and $\{z, z + z_j, z + 2z_j, \ldots, z + nz_j\}$.

The first sum converges to $h(\omega)$, the last one to $h(T_z \omega)$. By the pointwise ergodic theorem the first sum on the right converges to 0 because it has only n^{r-1} terms. Consequently all terms converge a.s. The second sum on the left must also vanish in the limit because it converges to zero in probability. We get $h(\omega) = h(T_z \omega)$ $\forall z \in \mathcal{R}$ and conclude by ergodicity and the mean-zero property of F that $h = 0$. Then (A.8) follows by a Riemann sum-type approximation. $\square$

PROOF OF THEOREM A.4. This goes by induction on r. For $r = 1$, rearrange

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{f(\omega, kz_1)}{n} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{n} \sum_{i=0}^{k-1} F(T_{iz_1} \omega, z_1) = \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k+1}{n}\right) F(T_{kz_1} \omega, z_1).$$

An application of (A.8) with $g(y) = 1 - y$ gives conclusion (A.7) for $r = 1$.

Suppose that (A.7) holds for some $r \in \{1, \ldots, M-1\}$. Let us show it for $r + 1$.

$$\frac{1}{n^{r+1}} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \sum_{k_{r+1}=0}^{n-1} \frac{f(\omega, k_1 z_1 + \cdots + k_r z_r + k_{r+1} z_{r+1})}{n}$$

$$= \frac{1}{n^{r+2}} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \sum_{k_{r+1}=0}^{n-1} \bigl[f(\omega, k_1 z_1 + \cdots + k_r z_r)$$

$$+ f(T_{k_1 z_1 + \cdots + k_r z_r} \omega, k_{r+1} z_{r+1}) \bigr]$$

$$= \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \frac{f(\omega, k_1 z_1 + \cdots + k_r z_r)}{n}$$

$$+ \frac{1}{n^{r+2}} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{r+1}=0}^{n-1} f(T_{k_1 z_1 + \cdots + k_r z_r} \omega, k_{r+1} z_{r+1})$$

$$= \frac{1}{n^r} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \frac{f(\omega, k_1 z_1 + \cdots + k_r z_r)}{n}$$

$$+ \frac{1}{n^{r+1}} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{r+1}=0}^{n-1} \left(1 - \frac{k_{r+1}+1}{n}\right) F(T_{k_1 z_1 + \cdots + k_{r+1} z_{r+1}} \omega, z_{r+1}).$$

As $n \to \infty$ on the last line, the first sum goes to zero by the induction hypothesis and the second sum by (A.8) with $g(y_1, \dots, y_{r+1}) = 1 - y_{r+1}$. $\square$

PROOF OF THEOREM A.3. Fix a labeling $z_1, \dots, z_M$ of the steps in $\mathcal{R}$. We first prove

$$\varliminf_{n\to\infty} \min_{x\in D_n} \frac{f(\omega, x)}{n} \ge 0. \tag{A.9}$$

Let $\delta > 0$ and $a_k = k\delta/(4M)$ for $k \in \mathbb{Z}_+$. For $\mathbf{k} = (k_1, \dots, k_M) \in \mathbb{Z}_+^M$ define sets

$$B_{n,\mathbf{k}} = \left\{ \sum_{i=1}^{M} s_i z_i : \lfloor n a_{k_i} \rfloor \le s_i < \lfloor n a_{k_i+1} \rfloor \text{ for } i \in [M] \right\}.$$

For each $x \in D_n$ we can pick $B_{n,x} = B_{n,\mathbf{k}(x)}$ such that every point $y \in B_{n,x}$ can be reached from x with an admissible path of at most $n\delta$ steps. (The assumption $x \in D_n$ implies $x = \sum_{i=1}^M b_i z_i$ with $\sum_{i=1}^M b_i = n$. For each i take k_i minimal such that $\lfloor n a_{k_i} \rfloor \ge b_i$.) Our strategy is to replace $f(\omega, x)$ with an average of f over $B_{n,x}$. Note that there is a fixed finite set K of vectors $\mathbf{k}$ such that the above choices can be made from $\{B_{n,\mathbf{k}} : \mathbf{k} \in K\}$ for all large enough n and all $x \in D_n$.

For every $x \in D_n$ and every $y \in B_{n,x}$ fix a path from x to y such that the steps $z_1, z_2, \dots, z_M$ are taken in order. Recall that $F(\omega, 0) = 0$. Then for any such pair x, y, with designated path $(x_i)_{i=0}^m$,

$$\begin{aligned} f(\omega, x) &= f(\omega, y) - \sum_{i=0}^{m-1} F(T_{x_i}\omega, x_{i+1} - x_i)\mathbf{1}\{x_{i+1} \ne x_i\} \\ &\ge f(\omega, y) - \sum_{i=0}^{m-1} \overline{F}(T_{x_i}\omega, x_{i+1} - x_i)\mathbf{1}\{x_{i+1} \ne x_i\} \\ &\ge f(\omega, y) - \sum_{z\in\mathcal{R}\setminus\{0\}} \left\{ \max_{|u|_1 \le 2nr_0} \sum_{0\le i\le n\delta} \left|\overline{F}(T_{u+iz}\omega, z)\right| \right\}. \end{aligned}$$

Above $r_0 = \max\{|z|_1 : z \in \mathcal{R}\}$. The error term is independent of x, y. Average over $y \in B_{n,x}$, and then take minimum over $x \in D_n$,

$$\begin{aligned} \min_{x\in D_n} \frac{f(\omega, x)}{n} \ge{} & \min_{\mathbf{k}\in K} \frac{1}{N_{n,\mathbf{k}}} \sum_{s_1=\lfloor n a_{k_1} \rfloor}^{\lfloor n a_{k_1+1} \rfloor - 1} \cdots \sum_{s_M=\lfloor n a_{k_M} \rfloor}^{\lfloor n a_{k_M+1} \rfloor - 1} \frac{f(\omega, s_1 z_1 + \cdots + s_M z_M)}{n} \\ & - \sum_{z\in\mathcal{R}\setminus\{0\}} \left\{ \max_{|u|_1 \le 2nr_0} \frac{1}{n} \sum_{0\le i\le n\delta} \left|\overline{F}(T_{u+iz}\omega, z)\right| \right\}, \end{aligned}$$

where $N_{n,\mathbf{k}} = \prod_{i=1}^M (\lfloor n a_{k_i+1} \rfloor - \lfloor n a_{k_i} \rfloor) \sim C n^M$. As $n \to \infty$, the first term on the right vanishes by Theorem A.4. After that let $\delta \to 0$, and assumption (A.5) takes care of the last term. Bound (A.9) has been verified.

To prove

$$\overline{\lim_{n\to\infty}} \max_{x\in D_n} \frac{f(\omega,x)}{n} \le 0, \tag{A.10}$$

we repeat the argument but with more rectangles.

For $\varnothing \ne I \subset [M]$ and $\mathbf{k} = (k_i)_{i\in I} \subset \mathbb{Z}_+^{|I|}$, define

$$B_{n,I,\mathbf{k}} = \left\{ \sum_{i\in I} s_i z_i : \lfloor n a_{k_i} \rfloor \le s_i < \lfloor n a_{k_i+1} \rfloor \text{ for } i \in I \right\}.$$

For each $x \in D_n$ pick $B_{n,x} = B_{n,I(x),\mathbf{k}(x)}$ so that x can be reached from every point $y \in B_{n,x}$ with an admissible path of at most $n\delta$ steps. The additional flexibility of choice of $I(x)$ accommodates points $x = \sum_{i=1}^M b_i z_i$ such that some $b_i < \lfloor n a_1 \rfloor$ and therefore a rectangle $B_{n,\mathbf{k}}$ that uses all M steps cannot be placed "upstream" from x. As before, there is a fixed finite set K from which all the vectors $\mathbf{k}(x)$ can be chosen, for all $x \in D_n$ and large enough n.

For every $x \in D_n$ and $y \in B_{n,x}$ fix a path from y to x such that the steps z_j, $j \in I(x)$, are taken in order. Then for any such pair x, y, with designated path $(x_i)_{i=0}^m$,

$$\begin{aligned} f(\omega,x) &= f(\omega,y) + \sum_{i=0}^{m-1} F(T_{x_i}\omega, x_{i+1}-x_i)\mathbf{1}\{x_{i+1} \ne x_i\} \\ &\le f(\omega,y) + \sum_{i=0}^{m-1} \overline{F}(T_{x_i}\omega, x_{i+1}-x_i)\mathbf{1}\{x_{i+1} \ne x_i\} \\ &\le f(\omega,y) + \sum_{z\in\mathcal{R}\setminus\{0\}} \left\{ \max_{|u|_1 \le 2nr_0} \sum_{0\le i\le n\delta} \left|\overline{F}(T_{u+iz}\omega, z)\right| \right\}. \end{aligned}$$

Again average over $y \in B_{n,x}$ to obtain

$$\begin{aligned} &\max_{x\in D_n} \frac{f(\omega,x)}{n} \\ &\quad\le \max_{\substack{\mathbf{k}\in K \\ \varnothing\ne I\subset[M]}} \frac{1}{N_{n,I,\mathbf{k}}} \sum_{s_{j_1}=\lfloor n a_{k_{j_1}} \rfloor}^{\lfloor n a_{k_{j_1}+1} \rfloor -1} \cdots \sum_{s_{j_{|I|}}=\lfloor n a_{k_{j_{|I|}}} \rfloor}^{\lfloor n a_{k_{j_{|I|}}+1} \rfloor -1} \frac{f(\omega, s_{j_1} z_{j_1} + \cdots + s_{j_{|I|}} z_{j_{|I|}})}{n} \\ &\qquad + \sum_{z\in\mathcal{R}\setminus\{0\}} \left\{ \max_{|u|_1 \le 2nr_0} \frac{1}{n} \sum_{0\le i\le n\delta} \left|\overline{F}(T_{u+iz}\omega, z)\right| \right\}, \end{aligned}$$

where $N_{n,I,\mathbf{k}} \sim Cn^{|I|}$ and $I = \{j_1, \dots, j_{|I|}\}$. Bound (A.10) follows as above. □

REFERENCES

[1] BAKHTIN, Y., CATOR, E. and KHANIN, K. (2014). Space–time stationary solutions for the Burgers equation. *J. Amer. Math. Soc.* **27** 193–238. MR3110798

[2] BORODIN, A. and CORWIN, I. (2014). Macdonald processes. *Probab. Theory Related Fields* **158** 225–400. MR3152785

[3] CARMONA, P. and HU, Y. (2002). On the partition function of a directed polymer in a Gaussian random environment. *Probab. Theory Related Fields* **124** 431–457. MR1939654

[4] CATOR, E. and PIMENTEL, L. P. R. (2012). Busemann functions and equilibrium measures in last passage percolation models. *Probab. Theory Related Fields* **154** 89–125. MR2981418

[5] COMETS, F., SHIGA, T. and YOSHIDA, N. (2003). Directed polymers in a random environment: Path localization and strong disorder. *Bernoulli* **9** 705–723. MR1996276

[6] COMETS, F., SHIGA, T. and YOSHIDA, N. (2004). Probabilistic analysis of directed polymers in a random environment: A review. In *Stochastic Analysis on Large Scale Interacting Systems*. *Adv. Stud. Pure Math.* **39** 115–142. Math. Soc. Japan, Tokyo. MR2073332

[7] COMETS, F. and VARGAS, V. (2006). Majorizing multiplicative cascades for directed polymers in random media. *ALEA Lat. Am. J. Probab. Math. Stat.* **2** 267–277. MR2249671

[8] COMETS, F. and YOSHIDA, N. (2006). Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.* **34** 1746–1770. MR2271480

[9] CORWIN, I. (2012). The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl.* **1** 1130001, 76. MR2930377

[10] CORWIN, I., O'CONNELL, N., SEPPÄLÄINEN, T. and ZYGOURAS, N. (2014). Tropical combinatorics and Whittaker functions. *Duke Math. J.* **163** 513–563. MR3165422

[11] DAMRON, M. and HANSON, J. (2014). Busemann functions and infinite geodesics in two-dimensional first-passage percolation. *Comm. Math. Phys.* **325** 917–963. MR3152744

[12] DEN HOLLANDER, F. (2009). *Random Polymers*. *Lecture Notes in Math.* **1974**. Springer, Berlin. MR2504175

[13] FERRARI, P. A., MARTIN, J. B. and PIMENTEL, L. P. R. (2009). A phase transition for competition interfaces. *Ann. Appl. Probab.* **19** 281–317. MR2498679

[14] FERRARI, P. A. and PIMENTEL, L. P. R. (2005). Competition interfaces and second class particles. *Ann. Probab.* **33** 1235–1254. MR2150188

[15] GEORGIOU, N. and SEPPÄLÄINEN, T. (2013). Large deviation rate functions for the partition function in a log-gamma distributed random potential. *Ann. Probab.* **41** 4248–4286. MR3161474

[16] HOFFMAN, C. (2005). Coexistence for Richardson type competing spatial growth models. *Ann. Appl. Probab.* **15** 739–747. MR2114988

[17] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** 437–476. MR1737991

[18] KOSYGINA, E. and VARADHAN, S. R. S. (2008). Homogenization of Hamilton–Jacobi–Bellman equations with respect to time–space shifts in a stationary ergodic medium. *Comm. Pure Appl. Math.* **61** 816–847. MR2400607

[19] KRENGEL, U. (1985). *Ergodic Theorems*. *De Gruyter Studies in Mathematics* **6**. Walter de Gruyter, Berlin. With a supplement by Antoine Brunel. MR0797411

[20] LACOIN, H. (2010). New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. Math. Phys.* **294** 471–503. MR2579463

[21] NEWMAN, C. M. (1995). A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians*, *Vol.* 1, 2 (*Zürich*, 1994) 1017–1023. Birkhäuser, Basel. MR1404001

[22] O'CONNELL, N., SEPPÄLÄINEN, T. and ZYGOURAS, N. (2014). Geometric RSK correspondence, Whittaker functions and symmetrized random polymers. *Invent. Math.* **197** 361–416.

[23] O'CONNELL, N. and YOR, M. (2001). Brownian analogues of Burke's theorem. *Stochastic Process. Appl.* **96** 285–304. MR1865759

[24] QUASTEL, J. (2010). Weakly asymmetric exclusion and KPZ. In *Proceedings of the International Congress of Mathematicians*. *Volume IV* 2310–2324. Hindustan Book Agency, New Delhi. MR2827973

[25] RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2014). Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields* **158** 711–750. MR3176363

[26] RASSOUL-AGHA, F., SEPPÄLÄINEN, T. and YILMAZ, A. (2013). Quenched free energy and large deviations for random walks in random potentials. *Comm*. *Pure Appl*. *Math*. **66** 202–244. MR2999296

[27] ROSENBLATT, M. (1971). *Markov Processes*. *Structure and Asymptotic Behavior*. *Die Grundlehren der mathematischen Wissenschaften* **184** Springer, New York. MR0329037

[28] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann*. *Probab*. **40** 19–73. MR2917766

[29] SPOHN, H. (2012). Stochastic integrability and the KPZ equation. Available at arXiv:1204.2657.

[30] TRACY, C. A. and WIDOM, H. (2002). Distribution functions for largest eigenvalues and their applications. In *Proceedings of the International Congress of Mathematicians*, *Vol*. *I* (*Beijing*, 2002) 587–596. Higher Ed. Press, Beijing. MR1989209

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