MATH 5050: Homework 4 (Due Monday, April 14)

Problem 1 Consider the heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + 3\frac{\partial u}{\partial x}(t,x) \text{ for } t > 0 \text{ and all } x, \tag{1}$$

with initial heat profile $u(0,x) = e^{-x^2}$. Similarly to how we solved the heat equation without the $\frac{\partial u}{\partial x}$ term, one can solve this equation using a standard Brownian motion B as follows:

$$u(t,x) = E[e^{-(x+\sigma B(t)+\mu t)^2}].$$

Here, σ and μ need to be chosen appropriately, based on the coefficients in (1).

(a) We want to get an explicit formula for u. For this, first compute

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(az+b)^2} e^{-z^2/2}\, dz.$$

Hint: This is a calculus question. Expand the squares, combine the z^2 terms, complete the square, and conclude by observing that you are led to a constant times the integral of the pdf of a normal.

(b) Write $u(t,x)$ as an integral, using the fact that $B(t)$ is a normal random variable with mean 0 and variance t and therefore has the same distribution as $\sqrt{t}Z$, where Z is a standard normal. Then by making an appropriate choice of the numbers a and b in part (a), find an explicit formula for $u(t,x)$.

(c) Plug in your formula for u into (1) and figure out the values of σ and μ.

(d) Plot $u(0,x)$.

(e) Let $t = 0.1$. What is the distribution of $B(0.1)$? Numerically compute and then plot $u(0.1,x)$, for $x \in [-20, 20]$.

Hint: You do not need to generate a whole path of Brownian motion. Just need to generate many (say $10,000$) samples of the variable $B = B(0.1)$. Then calculate the sample mean of $e^{-(x+\sigma B+0.1\mu)^2}$ to get the approximate value of $u(0.1,x)$ for x going from -20 to 20 (with say increments of 0.01). [Make sure the mean is over the samples, not over the x's!]

(f) On the same graph, plot $u(0.1,x)$ based on the formula you found in (b). This should agree with the curve you plotted in (e).

(g) Repeat (e) and (f) for $t = 1$ and $t = 10$. Do plots look like what you expect u should do? Explain.

Problem 2 Let $B(t)$ be standard Brownian motion. Let $Y(t)$ take the value 1 if $0 \le t < 2$ and the value $3B(2)$ if $t \ge 2$. Prove that

$$M(t) = \int_0^t Y(s)\, dB(s)$$

is a martingale relative to the filtration given by $\mathcal{F}_t = \{B(s) : s \le t\}$. (You need to do the proof directly. You cannot just say that it is a martingale because it is a stochastic integral of a progressively measurable function.)

Hint: $M(t)$ can be easily computed. Consider the cases $t < 2$ and $t \ge 2$ separately. After that you need to prove that for $s < t$ we have $E[M(t)|\mathcal{F}_s] = M(s)$. For this, consider the cases $s < t < 2$, $2 \le s < t$, and $s < 2 \le t$ separately.

Problem 3 Let $B(t)$ be standard Brownian motion.

(a) Calculate numerically $\int_0^t B(s)\, dB(s)$ for $0 \le t \le 10$. Use the leftmost points of the partition intervals and then the rightmost points. Plot both processes (using the same sample of Brownian motion, not two independent samples).

(b) Based on the plots, which of the two is the correct approximation of the stochastic integral and why?

(c) Calculate $\int_0^t B(s)\, dB(s)$ analytically (not numerically).

Hint: Use Itō's formula:

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(s))\, dB(s) + \frac{1}{2}\int_0^t f''(B(s))\, ds \tag{2}$$

with $f(x) = x^2$.

(d) If you use the same sample of Brownian motion that you used in part (a) to plot the explicit solution you got in (c), then the latter should match one of the former two. Is this the case? Does this agree with your answer to (b)?

Problem 4 Let $B(t)$ be standard Brownian motion. Let λ be a positive constant and let $X(t) = e^{\lambda B(t)}$.

(a) Apply Itō's formula (2) to the function $f(x) = e^{\lambda x}$ to prove that $X(t)$ solves

$$X(t) = 1 + \frac{\lambda^2}{2}\int_0^t X(s)\, ds + \lambda \int_0^t X(s)\, dB(s).$$

(b) Let $A(t) = E[X(t)]$. Observe that $A(0) = 1$. Using the fact that a stochastic integral is a martingale prove that $A(t) = 1 + \frac{\lambda^2}{2}\int_0^t A(s)\, ds$.

(c) The equation in (b) says that $A'(t)/A(t) = \frac{\lambda^2}{2}$. Integrate both sides (and use $A(0) = 1$) to find the formula for $A(t)$.

(d) What is the distribution of $B(t)$? Can you compute $E[e^{\lambda B(t)}]$ directly? Does your answer agree with (c)?

Hint: Recall moment generating functions!

Recall the following variant of Itō's formula: Let $f(t,x)$ have two continuous derivatives in x and one continuous derivative in t, and let $Z(t) = f(t, B(t))$. Let $\partial_t f$ denote the derivative in the t variable and $\partial_x f$ and $\partial xx f$ the first and second derivatives in the x variable. Then a Taylor expansion (and ignoring texrms that have something smaller than ds) gives

$$Z(t) - Z(0) = \int_0^t \partial_t f(s, B(s))\, ds + \int_0^t \partial_x f(s, B(s))\, dB(s) + \frac{1}{2} \int_0^t \partial_{xx} f(s, B(s))\, ds\,.$$

Problem 5 Let $B(t)$ be a standard Brownian motion. Prove that $M(t) = e^{\lambda B(t) - \lambda^2 t/2}$ is a martingale in the filtration of Brownian motion $\mathcal{F}_t = \{B(s) : s \leq t\}$.

Hint: Apply Itō's formula with $f(t,x) = e^{\lambda x - \lambda^2 t/2}$ to prove that $M(t)$ is equal to a stochastic integral. An alternate way would be by computing $E[M(t) \,|\, B(s)]$ using moment generating functions. It is a good exercise that way too, but it does not let you practice your Itō's formula skills! So maybe it is best to do it both ways!