MATH 5050: HOMEWORK 2 (DUE MONDAY, FEB. 17)

Problem 1 Fix $0 < p < 1$ with $p \neq 1/2$. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables taking values 1 and -1 with probability p and $1-p$, respectively. Let $S_n = X_1 + \cdots + X_n$. This is precisely a random walk that starts at 0 and then moves left or right one step each time with probability $1-p$ and p, respectively.

(a) Prove that $M_n = \left(\frac{1-p}{p}\right)^{S_n}$ is a martingale in the filtration $\mathcal{F}_n$ generated by $X_1, \ldots, X_n$.

(b) Fix integers $a < 0 < b$. Let T be the first time the walk S_n reaches the set $\{a, b\}$:

$$T = \inf\{n \geq 0 : S_n = a \text{ or } S_n = b\}.$$

If the walk never reaches $\{a, b\}$, then we set $T = \infty$. Prove that $P\{T < \infty\} = 1$, $E[|M_T|] < \infty$, and that $E[M_T] = 1$.

(c) Use the above to calculate the probability that starting at 0 the walk will reach a before it reaches b.

(d) Calculate the two probabilities: starting at 0 the walk ever reaches -1 and starting at 0 the walk ever reaches 1. When is this random walk transient and when is it recurrent?

Problem 2 Fix $0 < p < 1$ with $p \neq 1/2$. Let S_n be the random walk from Problem 1.

(a) Prove that $\overline{M}_n = S_n - (2p-1)n$ is a martingale in the filtration $\mathcal{F}_n$ generated by $X_1, \ldots, X_n$.

(b) Fix integers $a < 0 < b$. Consider the stoping time T from part (b) of the previous problem. You have proved that $P\{T < \infty\} = 1$. Now prove that $E[|\overline{M}_T|] < \infty$ and that $E[\overline{M}_T] = 0$.

(c) Calculate $E[T]$.

(d) Calculate the average time it takes the walk to reach -1, starting at 0.

Problem 3 Let Z_n be the branching process defined as follows. For $n \geq 0$, Z_n is the number of individuals in generation n. $Z_0 = 1$ (one individual in generation 0). At generation $n \geq 0$ each of the Z_n individuals, independently of all the others, gives birth to k new individual with probability p_k. Of course $\sum_{k=0}^{\infty} p_k = 1$. Each individual dies immediately after having given birth.

Let $\mu = \sum_{k=0}^{\infty} kp_k$ (the average number of individuals born from one individual in the previous generation).

(a) Is Z_n a martingale in the filtration $\mathcal{F}_n$ generated by $Z_1, \ldots, Z_n$? Why or why not?

(b) Prove that $M_n = Z_n/\mu^n$ is a martingale in the filtration $\mathcal{F}_n$ generated by $Z_1, \ldots, Z_n$.