MATH 5050/6815: Homework 1 (Due Monday, January 27)

**Problem 1.** Write a code (in your programming language of preference) that does the following:

a) Generates $n$ independent exponential random variables with mean $\mu$ (i.e. rate $\lambda = 1/\mu$). Here, $n$ and $\mu$ are parameters that you specify at the beginning of the code. For this particular homework, set $n = 10,000$ and $\mu = 2$.

b) Given a threshold value $s$ (specified at the beginning of the code), the code picks out all the outcomes from part a) that are $\geq s$, subtracts $s$ from each value, and plots a normalized histogram of the resulting data. For this particular homework, set $s = 1$. E.g. if we the first six values were 0.8234, 1.2342, 1.1223, 0.9298, 3.2394, 2.2345, then the values 0.8234 and 0.9298 are ignored and the new data is 1.2342-1=0.2342, 1.1223-1=0.1223, 3.2394-1=2.2394, and 2.2345-1=1.2345. Note: a normalized histogram is one where the column heights are divided by the total number of data points going into the histogram, so that the total area equals one.

c) On the same plot, draw the pdf of an exponential random variable with the given mean. Compare the histogram to the pdf.

*Remark: The above demonstrates the memoryless property, which says that conditioning an exponential random variable on being larger than a given value and then subtracting that value gives a random variable that has the same exponential distribution (with the same mean).*

**Problem 2.** Let $T_1$ and $T_2$ be two independent random variables that are both exponentially distributed with parameter $\lambda > 0$.

(a) What is the distribution of $T_1 + T_2$? What is its pdf?
(b) For $\varepsilon > 0$, calculate $\mathrm{P}(T_1 + T_2 \leq \varepsilon)$. *Hint:* Set up the integral and then use integration by parts.
(c) Show that $\varepsilon^{-1}\mathrm{P}(T_1 + T_2 \leq \varepsilon)$ goes to 0 as $\varepsilon \to 0$.

*Remark: The above shows that the probability a Poisson process has two or more jumps in a small amount of time $\varepsilon$ is of size much smaller than $\varepsilon$. It justifies why we neglected the process having two jumps or more when we were calculating the derivative of* $\mathrm{P}(N(t) = n)$.

**Problem 3.** Consider a Poisson process $N(t)$ with rate $\lambda = 1/2$. We want to calculate $\mathrm{P}(1 \leq N(4) \leq 3)$.

(a) Write a code to simulate the Poisson process using inter-arrival times $T_1, T_2, T_3, T_4$. Plot five samples of this process on the same graph. (Notice that this way of generating the process will go up to the <u>random</u> time $T_1 + T_2 + T_3 + T_4$ and so each sample will go up to a different time.)
(b) Write a code to simulate $N(4)$ by sampling a Poisson random variable with the appropriate parameter.
(c) Write a code to simulate the process up to time 4 by first sampling $N(4)$ as in (b) and then sampling the locations of the jumps between times 0 and 4. Plot five

samples of the process on the same graph. (This time, all five plots will be going up to the same time $t = 4$.)

(d) Generate 1000 samples of $N(4)$ using (b) and use them to estimate $\mathrm{P}(1 \leq N(4) \leq 3)$.

(e) Notice that $1 \leq N(4) \leq 3$ is the same exact thing as saying that $T_1 \leq 4 < T_1 + T_2 + T_3 + T_4$, i.e. that the first jump happened before time 4 and the fourth jump happened after time 4. Generate 10,000 samples of the process using (a) and use that to estimate $\mathrm{P}(1 \leq N(4) \leq 3)$.

(f) Compute the exact value of $\mathrm{P}(1 \leq N(4) \leq 3)$.

(g) Compare your answers to (d), (e), and (f).

**Problem 4.** Suppose that the lifetime of a component $T_i$ in hundreds of hours is uniformly distributed in [3,6]. Components are being replaced as soon as one fails and immediately with no time lost. Let $N_t$ be the number of components we changed by time $t$. You were away and came back at time $t = 100$ (i.e. after 10,000 hours of operation).

(a) Estimate the number of components used until now.

(b) Estimate the probability there were between 22 and 24 components used by now.

(c) Estimate the probability the current component has been in operation for at least 90 hours.

(d) Estimate the probability the current component will last for at least 500 more hours.

(e) Estimate the probability the total lifetime of the current component will be less than 450 hours.