(+1)

Lec. 26

Chapter 3

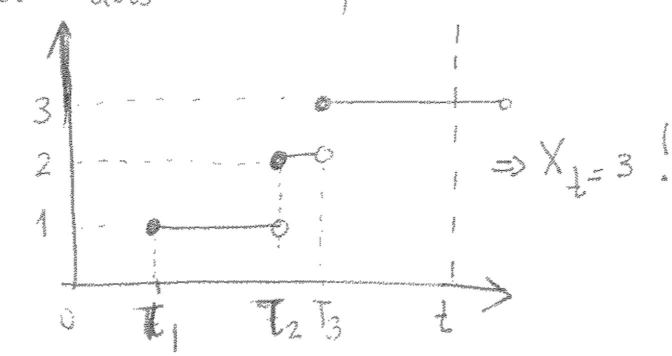
Continuous time Markov Chains.

The process $\{X_t\}_{t \in \mathbb{R}_+}$ is indexed by $\mathbb{R}_+$, uncountably many r.v.s involved!

The fact that time is continuous, allows us to model queing systems For example, let

$$X_t = \#\{\text{people entering Smith's by time } t\}$$

How does this process behave?



This is a "counting process" At some times (random) $T_1, T_2, T_n, \ldots$, it decides to jump by 1, signifying that customer entered the store.

$\boxed{Q:}$ Can two customers enter at exactly the same time?

$\boxed{A!}$ Some reasonable assumptions

1. Number of customers arriving during one time interval cannot affect the number of customers arriving during a different time interval.

2. The "average" rate at which customers arrive is constant

3. Customers arrive one at a time!

Mathematical version of assumptions.

) Let $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n \cdots$, $n \in \mathbb{N}$

Then the r.v.'s

$$X_{t_1} - X_{s_1}, \; X_{t_2} - X_{s_2}, \; \ldots, \; X_{t_n} - X_{s_n} \quad \text{are}$$

independent.

). If the arrival rate is fixed and $= \lambda$, then in an interval of size t, we expect λt customers

. In a small, tiny interval $(t, t+\Delta t)$, expect $\lambda \Delta t$ customers.

Assume $\lambda \Delta t << 1$ (can be done if Δt is really small).

$\mathbb{P}\{X_{t+\Delta t} = X_t\} = \mathbb{P}\{\text{no customer arrived in } (t, t+\Delta t)\} = 1 - \lambda \Delta t + o(\Delta t)$

$\mathbb{P}\{X_{t+\Delta t} = X_t + 1\} = \mathbb{P}\{1 \text{ customer arrived in } (t, t+\Delta t)\} = \lambda \Delta t + o(\Delta t)$

$\mathbb{P}\{X_{t+\Delta t} \geq X_t + 2\} = o(\Delta t)$

<u>Notation</u> $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$

stochastic process with $X_0 = 0$, X_t satisfying 1), 2) 3) is called poisson process with parameter λ.

$\mathbb{P}\{X_t = k\} = ?$

Let n sufficiently large (we'll decide later how large)

Write $$X_t = \sum_{j=1}^{n} \underbrace{\left[X_{jt/n} - X_{(j-1)t/n}\right]}$$ (recall $X_0 = 0$)

Independent (why?) (1)

Identically distributed (why?) (2)

nd from (3)

$\{X_{jt/n} - X_{(j-1)t/n} \geq 2$ for some $j \leq n\}$

$$\leq \sum_{j=1}^{n} \mathbb{P}\{X_{jt/n} - X_{(j-1)t/n} \geq 2\}$$

$$= n \cdot \mathbb{P}\{X_{t/n} \geq 2\} = \frac{o(t/n)}{1/n} \longrightarrow 0$$ as n grows.

Then the i.i.d r.v. $X_{jt/n} - X_{(j-1)t/n}$ are 0, 1 valued (Bernoulli) with prob. of success $\lambda \frac{t}{n}$ (why?) and their sum is Binomial

$\mathrm{Bin}\left(n, \frac{\lambda t}{n}\right) = Y$

becomes better as n grows!

$$\mathbb{P}\{X_t = k\} \approx \mathbb{P}\{Y = k\} = \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

et $n \to +\infty$ $$\mathbb{P}\{X_t = k\} = \lim_{n\to\infty} \frac{n!}{(n-k)!} \frac{1}{k!} \frac{(\lambda t)^k}{n^k} \cdot \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

$$= \frac{(\lambda t)^k}{k!} \lim_{n\to\infty} \underbrace{\frac{n!}{(n-k)!} \cdot \frac{1}{n^k}}_{1} \lim_{n\to\infty} \underbrace{\left(1 - \frac{\lambda t}{n}\right)^{n-k}}_{e^{-\lambda t}}$$

$$= e^{-\lambda t} (\lambda t)^k / k!$$

$$\Longrightarrow \quad \mathbb{P}\{X_t = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad \text{Poisson}(\lambda) \ \underline{\text{process}}$$

nother way to view Poisson process is with those event times/ aiting times T_i.

Waiting times

Let T_n to be the time between arrivals of $n-1$ and n customers

Let $Y_n = T_1 + T_2 + \dots + T_n$ total waiting time to see n-customers

So,
$$\boxed{Y_n = \inf\{t : X_t = n\}}$$
$$\boxed{T_n = Y_n - Y_{n-1}}$$

lly computation

$$\mathbb{P}\{T_1 > t\} \overset{!}{=} \mathbb{P}\{X_t = 0\} = e^{-\lambda t}.$$

$$\Longrightarrow \quad T_1 \sim \text{Exp}(\lambda) \quad , \mathbb{P}\{T_1 \leq s\} = 1 - e^{-\lambda s}.$$

Memoryless property of waiting times (~~Time homogeneity of Po~~ Shift invariance of Poisson)

For $i \geq 1$.

$$\mathbb{P}\{T_i > s+t \mid T_i > t\} = \mathbb{P}\{X_{s+t} \leq i-1 \mid X_t \leq i-1\}$$
$$= \mathbb{P}\{X_s \leq i-1 \mid X_0 \leq i-1\}$$
$$= \mathbb{P}\{X_s \leq i-1\}$$
$$= \mathbb{P}\{T_i > s\}.$$

$\mathbb{P}\{T_i > t\} = f(t)$.

Then $\mathbb{P}\{T_i > s+t \mid T_i > t\} = \mathbb{P}\{T_i > s\} \iff$

$\dfrac{\mathbb{P}\{T_i > s+t, T_i > t\}}{\mathbb{P}\{T_i > t\}} = \mathbb{P}\{T_i > s\} \iff$

$\boxed{f(s+t) = f(s)f(t)} \Rightarrow \boxed{e^{-\alpha t} = f(t)}$

Pf. $f(nt) = f(t)^n$

$f\left(\frac{t}{m}\right) = f(t)^{\frac{1}{m}}$

$f(\alpha t) = f(t)^{\alpha}$ $\alpha \in \mathbb{Q}_+$

$\exists \varepsilon > 0 : f(\varepsilon) > 0$

left. cont. $\Rightarrow f(t) = f(\varepsilon)^{\frac{t}{\varepsilon}} = e^{-\lambda t}$

$\mathbb{P}\{T_i > t\} = e^{-\alpha t}$ for some α.

~~But ...~~

T_i are identically distributed (does it make sense?) $\Rightarrow \alpha = \lambda$.

) T_i's are also independent. (why does this make sense?).

inite State space, continuous time MC.

- Exponential random variables.

Assume $T_1, \dots, T_n$ are exponential r.v.s with parameter (rates) $b_1, \dots, b_n$ respectively.

Done in Lec. 28

1) $T = \min\{T_1, T_2, \dots, T_n\}$ — T_i's are random alarm clocks, when will the first one go off?

$$\mathbb{P}\{T > t\} = \mathbb{P}\{\min\{T_1, \dots, T_n\} > t\}$$
$$= \mathbb{P}\{T_1 \geq t, T_2 \geq t, \dots, T_n \geq t\} = \mathbb{P}\{T_1 \geq t\}\mathbb{P}\{T_2 \geq t\} \cdots \mathbb{P}\{T_n \geq t\}$$
$$= e^{-b_1 t} e^{-b_2 t} \cdots e^{-b_n t} = e^{-(b_1 + \cdots + b_n)t}.$$

$\Rightarrow T \sim \mathcal{E}xp(b_1 + \dots + b_n)$ ■

2) Recall that continuous density can be loosely interpreted.

$$\mathbb{P}\{T_i = t\} \approx b_i e^{-b_i t} dt$$

Then,

$$\mathbb{P}\{T = T_i\} = \int_0^\infty \mathbb{P}\{T_1 > t, \dots, T_i = t, T_{i+1} > t, \dots, T_n > t\}$$

$$= \int_0^\infty e^{-b_1 t} e^{-b_2 t} \cdots e^{-b_{i-1} t} e^{-b_{i+1} t} \cdots e^{-b_n t} \, b_i e^{-b_i t} dt.$$

$$= \frac{b_i}{b_1 + \dots + b_n}.$$

ame result holds for infinite sequences as long as $\sum b_i < +\infty$.
(can also do it by verifying it for $n=2$, then using 1)) ■

<u>larkov property for continuous chains</u>! Back to Lec. 26

Let S be a finite state space. The process X_t is a Markov process on S iff for $t > s$

$$\boxed{\mathbb{P}\{X_t = y \mid X_r, 0 \le r \le s\} = \mathbb{P}\{X_t = y \mid X_s\}}$$

Furthermore, X_t is time homogeneous

$$\boxed{\mathbb{P}\{X_{t+s} = y \mid X_s = x\} = \mathbb{P}\{X_t = y \mid X_0 = x\}}$$

onstruction of a continuous time M.P.

- Define rates $\alpha(x,y) \geq 0$ for each pair $(x,y) \in S^2$, $x \neq y$.

So, if we are on state x, we jump to a state y with rate $\alpha(x,y)$ (See this as a certain proportion of time where you decide where to jump).

Define $\boxed{\alpha(x) = \sum_{y \neq x} \alpha(x,y)}$ to be the total rate at which the chain is changing from state x

A stochastic process that is time-homogeneous & Markov on S satisfies:

Assumption 1: $\mathbb{P}\{X_{t+\Delta t} = x \mid X_t = x\} = 1 - \alpha(x)\Delta t + o(\Delta t)$

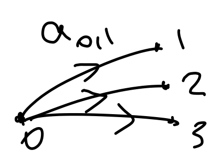
—ıı— 2: $\mathbb{P}\{X_{t+\Delta t} = y \mid X_t = x\} = \alpha(x,y)\Delta t + o(\Delta t)$

Heuristically:

On each site $y \in S$ attach independent exponential alarm clocks with rates $\alpha(x,y)$, for all $x \in S$.

Then the chain stays in state x until one of those alarms goes off and then, we jump at state y....

- This is Markovian - only the alarms involving the current state are important.
- It's time homogeneous (why?).



$T_{0,1} \sim \exp(a_{0,1})$
$T_{0,2} \sim \exp(a_{0,2})$
$T_{0,3} \sim \exp(a_{0,3})$

Say time now is 23.5. Say $T_{0,1} = 6.7$, $T_{0,2} = 5.1$, $T_{0,3} = 7.3$

The next jump is then to 2 at time $23.1 + 5.1 = 28.6$.

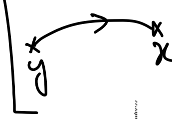
Define: $p_x(t) = \mathbb{P}\{X_t = x\}$ and use the limit definition of derivati

$$p_x'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\mathbb{P}\{X_{t+\Delta t} = x\} - \mathbb{P}\{X_t = x\}\right)$$

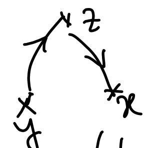
Now, $$\mathbb{P}\{X_{t+\Delta t} = x\} = \sum_{y \in S} \mathbb{P}\{X_{t+\Delta t} = x \mid X_t = y\} \cdot \mathbb{P}\{X_t = y\}$$

$$= \mathbb{P}\{X_{t+\Delta t} = x \mid X_t = x\} \cdot \mathbb{P}\{X_t = x\} + \sum_{y \neq x} \mathbb{P}\{X_{t+\Delta t} = x \mid X_t = y\} \cdot \mathbb{P}\{X_t = y\}$$

explanation:



in time Δt has prob. $\sim a_{y,x}\Delta t$



in time Δt has prob. $\sim (\Delta t)^2$ (two exp. clocks ringing b/w 0 & Δt)

$$p_x'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{-\mathbb{P}\{X_t = x\}\left(1 - \mathbb{P}\{X_{t+\Delta t} = x \mid X_t = x\}\right) + \sum_{y \neq x} \mathbb{P}\{X_{t+\Delta t} = x \mid X_t = y\}\mathbb{P}\{X_t = y\}\right\}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(-\alpha(x)\, p_x(t)\,\Delta t + o(\Delta t)\right) + \sum_{y \neq x} \lim_{\Delta t \to 0} \frac{1}{\Delta t}\, \alpha(y,x) \cdot p_y(t)\,\Delta t + o(\Delta t)$$

$$\boxed{p_x'(t) = -\alpha(x)\, p_x(t) + \sum_{y \neq x} \alpha(y,x)\, p_y(t)} \qquad (*)$$

ystem of differential equations

et $$A = a_{ij} = \begin{cases} \alpha(i,j) & i \neq j \\ -\alpha(i) & i = j \end{cases}, \quad i,j \in S$$

~~[illegible]~~. It's called the finitesimal generator!

en (*) can be written as $$\boxed{\bar{p}'(t) = \bar{p}(t) \cdot A}$$

Lec. 27

A has row sum $= 0$
non-positive diagonal entries
non-negative non diagonal entries
Always has a 0 eigenvalue!

In terms of transition matrices

Let $P_t(x,y) = \mathbb{P}\{X_t = y \mid X_0 = x\}$

$$P_t = \left(P_t(x,y) \right)_{x,y \in S}$$

Then, the differential equation can be written as

$$\frac{d}{dt} P_t = P_t A \quad , \quad P_0 = I$$

Then

$$P_t = e^{tA} \quad \text{(matrix exponential)}.$$

Also: $A = \lim_{t \to 0} \frac{P_t - I}{t}$
(So π inv. meas. $\Updownarrow$ $\pi A = 0$)

Definition: $$\boxed{e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}} \quad (*)$$

→ Homework problem (will be assigned): If $A = Q D Q^{-1}$ with D diagonal, show that $e^{tA} = Q e^{tD} Q^{-1}$.

Does $(*)$ always converge? Yes, all eigenvalues are ~~are~~ either 0, or have non-positive real part!

Example. Irreducible 2-state continuous MC

$$\begin{tikzcd} 0 \arrow[r, bend left, "\alpha(0,1)=1"] & 1 \arrow[l, bend left, "\alpha(1,0)=2"] \end{tikzcd} \Longrightarrow A = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \end{array}$$

summarizes $a(x,y)$'s. Rows add up to 0.

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \Rightarrow \text{High powers of } A \Rightarrow \text{diagonalize}$$

Eigenvalues: $\det\left(A - \lambda I\right) = 0 \Rightarrow \left|\begin{pmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{pmatrix}\right| = 0$

$$\Longleftrightarrow (1+\lambda)(2+\lambda) - 1\cdot 2 = 0$$

$$\lambda^2 + 3\lambda = 0 \Rightarrow \lambda = 0, -3. \Rightarrow D_A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$

0-right eigenv   -3-right eigen

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$\Rightarrow P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \sum_{n=0}^{\infty} Q \frac{t^n D^n}{n!} Q^{-1}$$

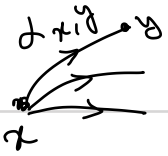
$$= Q \begin{bmatrix} 1 & 0 \\ 0 & \sum \frac{(-3t)^n}{n!} \end{bmatrix} Q^{-1} = Q \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} Q^{-1}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{bmatrix}$$

$$\lim_{t \to \infty} P_t = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}$$

What is πA ?

Lec. 28



first time a clock rings is again an exp. random varible w/ rate exactly $\alpha(x) = \sum_y \alpha(x,y)$ and probability the (x,y) bond rings is $\frac{\alpha(x,y)}{\alpha(x)}$.

So we can replace the original MC w/ an equivalent one that only has clocks on the sites: site x has an exp. clock with rate $\alpha(x)$. Whe the clock rings, it's time for a jump. MC jumps to y w/ prob. $\frac{\alpha(x,y)}{\alpha(x)}$.

This can even be done for a countable state space, as long as $\alpha(x) < \infty$.

If $a = \sup_x a(x) < \infty$ (always the case for a finite space), then we can even use one clock for all sites! The rate of this clock is a. When it rings, the chain moves, but now the moves are not necessarily to $y \neq x$. The chain could stay put at x and wait for the next ring. This is b/c if the rates $\alpha(x)$ and $\alpha(y)$ differ, then we cannot use the same rate @ x & y unless we allow for staying @ the one with the lower rate. This will be the subject of a HW exercise.

Lec. 29

Mean passage times

X_t is irreducible, continuous time MC, finite state space S

Define $T = \inf\{ t : X_t = z \}$

$$b(x) = \mathbb{E}(T \mid X_0 = x)$$

Then $b(z) = 0$ (Right?)

Define $\mathcal{S}$ to be the first time that the chain changes state

Then

$$\boxed{\mathbb{E}(T \mid X_0 = x) = \mathbb{E}(\mathcal{S} \mid X_0 = x) + \sum_{y \in S} \mathbb{P}\{X_{\mathcal{S}} = y \mid X_0 = x\} \, \mathbb{E}(T \mid X_0 = y)}$$

Proof

First note that $Y \geq T$ (why?) We can re-write

$$Y = T + \inf\{ t \geq T : X_t = z \}$$

$$= T + \inf\{ t \geq 0 : X_{T+t} = z \}$$

$$= T + \inf\{ t \geq 0 : X_{T+t} = z \} \sum_{y \in S} \mathbb{1}\{X_T = y\}$$

Then $\mathbb{E}(Y \mid X_0 = x) = \mathbb{E}(T \mid X_0 = x) + \sum_{y \in S} \mathbb{E}\left(\inf\{t \geq 0 : X_{T+t} = z\} \mathbb{1}\{X_T = y\} \mid X_0 = x \right)$

$$= \mathbb{E}(T \mid X_0 = x) + \sum_{y \in S} \mathbb{E}\left(\inf\{t \geq 0 : X_{T+t} = z\} \mid X_T = y \right) \mathbb{E}\left(\mathbb{1}\{X_T = y\} \mid X_0 \right)$$

$$= \mathbb{E}(T \mid X_0 = x) + \sum_{y \in S} \mathbb{P}\{X_T = y \mid X_0 = x\} \, \mathbb{E}[Y \mid X_0 = y]$$

Now, T is exponential with rate $\alpha(x)$ $\Longrightarrow$

$$E(S \mid X_0 = x) = \frac{1}{\alpha(x)}$$

$\mathbb{P}\{X_S = y \mid X_0 = x\} = \mathbb{P}\{$ the first alarm that rings was y's $\mid X_0 = x\}$

$$= \frac{\alpha(x,y)}{\alpha(x)} \qquad (\text{why?})$$

$b(z) = 0$

So the sum becomes

$$b(x) = \frac{1}{\alpha(x)} + \sum_{\substack{y \neq x \\ \neq z}} {\scriptstyle(\text{why?})} \frac{\alpha(x,y)}{\alpha(x)} \cdot b(y)$$

$$\Longrightarrow \quad \alpha(x)\, b(x) = 1 + \sum_{y \neq x, z} \frac{\alpha(x,y)}{\cancel{\alpha(x)}} \cdot b(y).$$

$$\Longrightarrow \quad 0 = 1 + \sum_{y \neq x, z} \alpha(x,y)\, b(y) - \alpha(x)\, b(x).$$

matrix form.

vector

$$\boxed{\bar{0} = \bar{1} + \tilde{A}\, \bar{b}}$$

$$\Updownarrow$$

$$\boxed{\bar{b} = [-\tilde{A}]^{-1}\, \bar{1}}$$

where $\tilde{A}$ is obtained by A by deleting the row and column associated to $\underline{z}$.

$\boxed{\text{Remember}}$ z is specified form the problem. Different z's mean different $\tilde{A}$ so, the procem must be repeated.

xample

$$A = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \end{array}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 & 1 & 0 & -1/3 \\ 1 & 0 & 0 & 1 \\ 1 & -1/2 & -1/2 & -1/3 \\ 1 & -1/2 & 1/2 & -1/3 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 2/3 & 0 & -1/3 & -1/3 \\ 0 & 0 & -1 & 1 \\ -1/4 & 3/4 & -1/4 & -1/4 \end{bmatrix}$$

$$\text{Find } P_t = \begin{bmatrix} 1/4 \end{bmatrix} = \frac{1}{4} \cdot 1 + e^{-t} \begin{bmatrix} 2/3 & 0 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 \\ -1/3 & 0 & 1/6 & 1/6 \\ '' & '' & '' & '' \end{bmatrix}$$

$$+ e^{-3t} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} + e^{-4t} \begin{bmatrix} 1/12 & -1/4 & 1/12 & 1/12 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ 1/12 & -1/4 & 1/12 & 1/12 \\ '' & '' & '' & '' \end{bmatrix}$$

ind π : Solve $\pi A = 0 \Rightarrow [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$

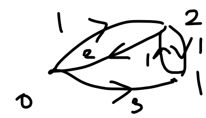
What are collumns sum of A ?

Compute the expected value of the time needed to get from 0 to 3.

· So $z = 3$! We want $b(0)$!

$$\tilde{A} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix} \end{array} \Rightarrow [-\tilde{A}]^{-1} = \begin{bmatrix} 5/3 & 2/3 & 1/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

example 2: 90



$$A = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} -4 & 3 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \end{array}$$

common rate : $\lambda = 4$

$$P = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

inv. meas.

$\pi A = 0$ or $\pi P = 0$ both give $\pi = [\frac{1}{8}\ \frac{5}{8}\ \frac{1}{4}]$

⚠ P_t has only > 0 e-val's ! So $\neq P$

time $0 \to 1$:

$$\tilde{A} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$$

$$(-\tilde{A})^{-1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}$$

time $0 \to 1$: $0.3 + 0.1 = 0.4$

Or : $Q = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$$(I - Q)^{-1} = \begin{bmatrix} 1.2 & 0.4 \\ 0.8 & 1.6 \end{bmatrix}$$

$0 \to 1$: $1.2 + 0.4 = 1.6$ steps

each takes $\frac{1}{4}$ time

so $\frac{1.6}{4} = 0.4$ time units

$\Rightarrow \vec{b} = [-\tilde{A}]^{-1}\vec{1} = (8/3, 5/3, 4/3)$

$\Rightarrow b(0) = 8/3, \quad b(1) = 5/3, \quad b(2) = 4/3$

Birth & Death Chain. Lec. 30

For this section: Small n denotes population

- Birth rates: λ_n, $n = 0, 1, 2, \ldots$
- Death rates: μ_n, $n = 0, 1, \ldots$

  $\mu_0 = 0$. (why?)

- Standing assumptions (changes of states: $n \to n+1$ or $n \to n-1$)

1. transitions

$$\mathbb{P}\{X_{t+\Delta t} = n+1 \mid X_t = n\} = \lambda_n \Delta t + o(\Delta t)$$

$$\mathbb{P}\{X_{t+\Delta t} = n-1 \mid X_t = n\} = \mu_n \Delta t + o(\Delta t)$$

$$\mathbb{P}\{X_{t+\Delta t} = n \mid X_t = n\} = 1 - (\mu_n + \lambda_n)\Delta t + o(\Delta t).$$

2. In a tiny interval, either a birth occurs (only) or a death occurs (only) or nothing at all!

Defining $\boxed{P_n(t) = \mathbb{P}\{X_t = n\}}$ we have a system of diff. eq's

$$\boxed{P_n'(t) = \mu_{n+1} P_{n+1}(t) + \lambda_{n-1} P_{n-1}(t) - (\mu_n + \lambda_n) P_n(t).}$$

(Try to prove this!)

xamples

Poisson process with rate parameter λ is birth and death with

$\lambda_n = \lambda$, $\mu_n = 0$ (How to prove this ?) Check equations!

Population models

ssume n is the number of individuals at a given time.
Each individual reproduces at rate λ and dies ~~with~~ rate μ.

Then $\lambda_n = n\lambda$, $\mu_n = n\mu$

- With immigration! $\lambda_n = n\lambda + \nu$, $\mu_n = n\mu$

- Fast growing model; pure birth!

  $\lambda_n = n^2\lambda$, $\mu_n = 0$

Define $\boxed{Y_n = \inf\{t : X_t = n\}}$ and let $Y_\infty = T_1 + T_2 + \cdots$

the time where the population becomes infinite! , T_i = time between i-th and i-th birth.

Then $\mathbb{E}(Y_\infty) = \sum_{i=1}^{\infty} \mathbb{E}(T_i) = \sum_{i=1}^{\infty} \frac{1}{i^2\lambda} < \infty$

$\Rightarrow$ $Y_\infty < \infty$ with probability 1

$\Rightarrow$ Explosion!

It always occurs when $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$! (iff statement!).

Embedded discrete chains / Transience / Null / Positive recurrence

Discrete time MC that mimics the continuous one when it moves!

$$p(n,n-1) = \frac{\mu_n}{\mu_n+\lambda_n}, \qquad p(n,n+1) = \frac{\lambda_n}{\mu_n+\lambda_n}.$$

<u>Lemma</u>. Continuous time B&D is recurrent iff discrete B&D is.

r discrete B&D's we know that however (First Lecture!)

<u>Thm</u> B&D are transient iff $\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$. $(*)$

<u>xample</u>

<u>ueing models</u>

Service rate $=\mu$, arrival rate $=\lambda$, 1 server M/M/1

$\Longrightarrow \lambda_n = \lambda, \quad \mu_n = \mu$

M/M/k Service rate $\mu_n = \begin{cases} n\mu & n \le k \\ k\mu & n > k \end{cases}$ $\quad \lambda_n = \lambda$.

M/M/∞: Service rate $\mu_n = n\mu$, arrival $\lambda_n = \lambda$.

<u>Then apply $(*)$ to decide recurrence and transience</u>

M/M/1 $\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^n$ $<\infty$ iff $\mu < \lambda \Leftrightarrow$ Transience; ∞ iff $\mu \geq \lambda \Leftrightarrow$ recurrence.

M/M/k. For all $\underset{\text{finite}}{n > k} \Rightarrow \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \frac{k!}{k^k}\left(\frac{k\mu}{\lambda}\right)^n$ $<\infty$ iff $k\mu < \lambda$; ∞ o.w.

* $M/M/\infty$, $\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} n! \left(\frac{\mu}{\lambda}\right)^n = \infty$ always! 94

Lec.31

Invariant distributions

Assume irreducible chains!

· Invariant distribution $\left\langle \begin{array}{l} \lim_{t\to\infty} \mathbb{P}\{X_t = n \mid X_0 = m\} = \pi(n) \\ \bar{\pi} \cdot \bar{P}_t = \bar{\pi} \\ \text{would be infinite!} \end{array} \right.$

If one exists, $\bar{\pi} \cdot \bar{P}_t = \bar{\pi} \Rightarrow \bar{\pi} \frac{d}{dt} P_t = 0 \; ! \Rightarrow \bar{\pi} A \cdot P_t = 0$

$(\Rightarrow \bar{\pi} A = 0)$.

Assume $P_t(n) = \mathbb{P}\{X_t = n\} = \pi(n)$

Then, from the diff. eq's of B&D chains,

$$\boxed{0 = \lambda_{n-1}\pi(n-1) + \mu_{n+1}\pi(n+1) - (\lambda_n + \mu_n)\pi(n).}$$

For $n = 0$, $\lambda_{-1} = 0 \Rightarrow \pi(1) = \frac{\lambda_0}{\mu_1}\pi(0)$

For $n \geq 1$,
$$\mu_{n+1}\pi(n+1) - \lambda_n \pi(n) = \mu_n \pi(n) - \lambda_{n-1}\pi(n-1)$$
$$= \mu_{n-1}\pi(n-1) - \lambda_{n-2}\pi(n-2)$$
$$\vdots$$
$$= \mu_1 \pi(1) - \lambda_0 \pi(0) = 0$$

$\Rightarrow \pi(n+1) = \frac{\lambda_n}{\mu_{n+1}}\pi(n) = \frac{\lambda_n \lambda_{n-1}}{\mu_{n+1}\mu_n}\pi(n-1) \cdots$

$$\Rightarrow \boxed{\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\pi(0).}$$

Then $\overline{\pi}$ can be made into a probability measure iff

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = q < \infty.$$

(

and

$$\boxed{\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \cdot q^{-1}}$$

So the chain is positive recurrent iff the sum converges

Example

M/M/1 queue!

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)^{-1} \quad \text{iff } \lambda < \mu.$$

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

Expected queue length in equilibrium:

$$\sum_{n=0}^{\infty} n\pi(n) = \sum_{n=0}^{\infty} n \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)^{-1} = \frac{\lambda}{\mu - \lambda}.$$

Class do M/M/∞ !

Population model: $\lambda_n = n\lambda$, $\mu_n = n\mu$

It's a branching process in cont. time!

When an individual is born, it does not die out in the next generation. It lives for a random time that is a $\exp(\mu)$.

While it lives, it keeps giving offsprings every $\exp(\lambda)$ time.

So, on average, during its life time of $\frac{1}{\mu}$ it will give $\frac{\lambda}{\mu}$ offsprings. I.e. what was μ for us in the discrete model is now $\frac{\lambda}{\mu}$. We expect exponential growth if $\frac{\lambda}{\mu} > 1$ and extinction if $\frac{\lambda}{\mu} \leq 1$. We expect exponential extinction if $\frac{\lambda}{\mu} < 1$.

Lec. 32

Let's see.

Transience if $\sum_{n=1}^{\infty} \frac{n!\,\mu^n}{n!\,\lambda^n} < \infty$ so if $\mu < \lambda$.

Positive rec. if $\sum_{n=1}^{\infty} \frac{(n-1)!\,\lambda^n}{n!\,\mu^n} < \infty$ so if $\mu > \lambda$. (To continue the chain after it reaches 0 we can say $\lambda_0 = \lambda$ instead of 0: $\lambda_0 = 0$)

$\lambda = \mu$: null rec. (we didn't prove this in the discrete case, but now we did!)

Let $f(t) = E[X_t \mid X_0 = 1] = \sum_{n=1}^{\infty} n P(X_t = n \mid X_0 = 1)$ (assuming here $\lambda_0 = 0$)

$$= \sum_{n=1}^{\infty} n P_t(1,n)$$

Recall: $\frac{d}{dt} P_t = P_t A$.

So $f'(t) = \sum_{n=1}^{\infty} n \sum_{k=0}^{\infty} p_t(1,k) A(k,n) = \sum_{n=1}^{\infty} n \Big((n-1)\lambda\, p_t(1,n-1) + (n+1)\mu\, p_t(1,n+1) - n(\lambda+\mu) p_t(1,n) \Big)$

$$= \lambda \sum_{n=2}^{\infty} n(n-1) p_t(1,n-1) + \mu \sum_{n=0}^{\infty} n(n+1) p_t(1,n+1) - (\lambda+\mu) \sum_{n=1}^{\infty} n^2 p_t(1,n)$$

$$= \sum_{n=1}^{\infty} \Big(\lambda(n+1)n + \mu(n-1)n - (\lambda+\mu) n^2 \Big) p_t(1,n)$$

$$= (\lambda-\mu) \sum_{n=1}^{\infty} n\, p_t(1,n) = (\lambda-\mu) f(t)$$

$f(0) = 1$ (why?)

So $f(t) = e^{(\lambda-\mu)t}$.

$\lambda > \mu \Rightarrow$ exp. growth

$\lambda < \mu \Rightarrow$ exp. decay

$\lambda = \mu \Rightarrow$ Population remains constant, on average

but extinction does happen 100% of the time!

Explanation: Null recurrence

(population has a good chance of getting huge before eventually getting extinct.)

Poisson Processes: $\lambda_n = \lambda$, $\mu_n = 0$ $\forall n \geq 0$.

We want $P(X_t = n \mid X_0 = 0) = P_t(0,n)$. Call this $g_n(t)$

$g_0'(t) = -\lambda g_0(t)$ (using $\frac{d}{dt} P_t = P_t A$ & that nothing goes to 0!)

$g_0(0) = 1$. So $g_0(t) = e^{-\lambda t}$.

for $n \geq 1$: $g_n'(t) = \overset{\text{rate from } n-1 \text{ to } n}{\lambda} g_{n-1}(t) - \overset{A(n,n) = -\lambda}{\lambda} g_n(t)$ and $g_n(0) = 0$

Let $f_n(t) = e^{\lambda t} g_n(t)$.

$$f_n'(t) = e^{\lambda t} g_n'(t) + \lambda e^{\lambda t} g_n(t) = \lambda e^{\lambda t} g_{n-1}(t) \cancel{- \lambda e^{\lambda t} g_n(t)} + \cancel{\lambda e^{\lambda t} g_n(t)}$$

So $f_n'(t) = \lambda f_{n-1}(t)$ and $f_n(0) = 0$.

$f_n''(t) = \lambda f_{n-1}(t) = \lambda^2 f_{n-2}(t)$ Lec. 33

$\overset{m \text{ derivatives}}{f_n^{(m)}}(t) = \lambda^m f_{n-m}(t)$ $0 \leq m \leq n$.

We see that $f_n^{(m)}(0) = 0$ for $0 \leq m \leq n-1$ and $n \geq 1$.

So $f_n^{(n)}(t) = \lambda^n f_0(t) = \lambda^n$. $f_n^{(n-1)}(t) = \lambda^n t$. $f_n^{(n-2)}(t) = \lambda^n \frac{t^2}{2}$

$\ldots$ $f_n(t) = \frac{\lambda^n t^n}{n!}$ and $g_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ $n \geq 1$

$n = 0$: same formula!

This shows that $X_t \sim \text{Poisson}(\lambda t)$

Ex. $P(\overset{\text{after 2 hours we got fewer than 2 jumps}}{X_2 \leq 1} \mid X_0 = 0) = P(X_2 = 1 \mid X_0 = 0) + P(X_2 = 0 \mid X_0 = 0) = \lambda t e^{-\lambda t} + e^{-\lambda t}$.

$$P(N_1 = 5, N_2 = 6 \mid N_2 = 6) = \frac{P(X_1 = 5, X_2 = 6)}{P(X_2 = 6)} = \frac{P(X_1 = 5) P(X_1 = 1)}{P(X_2 = 6)} = \frac{e^{-\lambda} \frac{\lambda^5}{5!} e^{-\lambda} \frac{\lambda}{1!}}{e^{-2\lambda} (2\lambda)^6 / 6!} = \frac{6!}{5! \times 1!} \frac{1}{2^6}$$

given 6 jumps happened in the first 2 hours, what is the prob. 5 of them happened in the first hour?

Binomial!