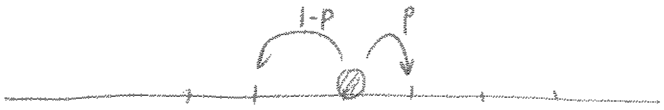
# Countable Markov Chains

# Lec. 15

xamples

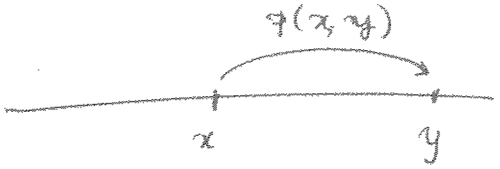
1) Random walk on $\mathbb{Z}$ (Simple)



$p(i,i+1) = p$

$p(i,i-1) = 1-p$ $\quad i \in \mathbb{Z}$.

) Random walk on $\mathbb{Z}$.



. Define $p(x,y) = \mathbb{P}\{X_1 = y \mid X_0 = x\}$

. $$\sum_{y\in S} p(x,y) = 1.$$

. $S = \mathbb{Z}$, $\Rightarrow$ infinite sum.

n-step transitions ; $P_n(x,y) = \mathbb{P}\{X_n = y \mid X_0 = x\}$

No matrix ?!.

## Chapman - Kolmogorov equations

If $0<m,n<\infty$

$$\begin{aligned} P_{m+n}(x,y) &= \mathbb{P}\{X_{m+n} = y \mid X_0 = x\} \\ &= \sum_{z\in\mathbb{Z}} \mathbb{P}\{X_{m+n} = y, X_m = z \mid X_0 = x\} \\ &= \sum_{z\in\mathbb{Z}} \mathbb{P}\{X_{m\underbrace{+n=y \mid X_m}_{n \qquad 0}} = z\}\, \mathbb{P}\{X_m = z \mid X_0 = x\} \\ &= \sum_{z\in\mathbb{Z}} P_m(x,z) \cdot P_n(z,y). \end{aligned}$$

$$\rightarrow \boxed{P_{m+n}(x,y) = \sum_{z\in\mathbb{Z}} P_m(x,z)\, P_n(z,y)}$$

This should make sense!

## 3) Random walk with partially reflecting boundary

$0<p<1$ , $S=\{0,1,2,\dots\}$

$p(x,x+1)=p$

$p(x,x-1)=1-p \qquad x\neq 0$

$p(0,0)=1-p$.

## Queing model (Discrete time)

Let $X_n$ = # of customers waiting in line for some service.

During each time interval, with prob. $p$ someone new arrives

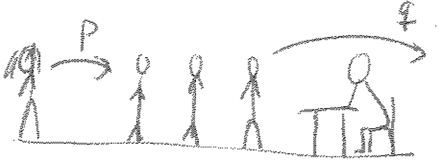
With probability $q$ someone gets serviced and leaves the queue.

- This is a Markov Chain. , $S=\{0,1,2,\dots\}$
- Transition probabilities:

$p(x,x-1) = q(1-p)$ (why?) $\quad p(x,x)= qp+(1-q)(1-p) \quad x\neq 0$

$p(x,x+1)= p(1-q) \quad x\neq 0$

$p(0,0)=1-p$ , $p(0,1)=p$



**Questions**: Can the line become infinite?

Is there a stationary distribution?

## Section 2.2. Recurrence and Transience.

n is an irreducible MC, countably infinite state space $S$ and ransition probabilities $p(x,y)$

Definitions:

The chain $X_n$ is recurrent if for each state $x \in S$,

$$\mathbb{P}\{X_n = x \text{ for infinitely many } n\} = 1$$

The chain $X_n$ is transient if for each state $x \in S$,

$$\mathbb{P}\{X_n = x \text{ for infinitely many } n\} = 0.$$

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Theorem.

An irreducible MC is transient if and only if the expected number of returns to a state $x$ is finite, i.e. iff

$$\boxed{\sum_{n=0}^{\infty} p_n(x,x) < \infty}, \quad \text{for some } x.$$

Proof. WLOG assume $X_0 = x$.

Let $\boxed{R_x = \sum_{n=0}^{\infty} \mathbb{1}\{X_n = x\}}$ = # of total visits to state $x$.

If $R_x \neq \infty$ then the chain cannot be recurrent, so it's transient.

But if $R_x < \infty$ then we can actually compute it's expectation,

$$\mathbb{E}(R_x) = \mathbb{E}\Big(\sum_{n=0}^{\infty} \mathbb{1}\{X_n = x\}\Big) = \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{1}\{X_n = x\})$$

$$= \sum_{n=0}^{\infty} \mathbb{P}\{X_n = x\} \overset{!}{\underset{\text{why?}}{=}} \sum_{n=0}^{\infty} p_n(x,x).$$

Now define

$$\boxed{T_x^1 = \min\{n > 0 : X_n = x\}}, \text{ the time of first return to } x.$$

We say $T_x = \infty$ if the chain does not return to $x$.

Assume by way of contradiction that $\boxed{\mathbb{P}\{T_x^1 < \infty\} = 1}$ (*)

(so we always return).

Recursively, define the time of k-th return

$$\boxed{T_x^k = \min\{n > T_x^{k-1} : X_n = x\} \quad , k \geq 2}$$

We discussed before how $T_x^k - T_x^{k-1} \overset{\mathcal{D}}{=} T_x^1$ and that

$T_x^k - T_x^{k-1} \perp T_x^j - T_x^{j-1} \quad \forall j \neq k$.

Thus, $\boxed{\mathbb{P}\{T_x^k - T_x^{k-1} < \infty\} = 1} \quad \forall k$.

But, $\mathbb{P}\{$we visit state $x$ infinitely many times$\} =$

$$= \lim_{k\to\infty} \mathbb{P}\{\text{we visit state } x \ k \text{ times}\}$$

$$= \lim_{k\to\infty} \mathbb{P}\{T_x^k < \infty\} = \lim_{k\to\infty} \mathbb{P}\{\underbrace{T_x^k - T_x^{k-1}}_{} + \underbrace{T_x^{k-1} - T_x^{k-2}}_{} + \dots + \underbrace{T_x^1}_{} < \infty\}$$

$$\underset{\text{why?}}{\overset{!}{=}} \lim_{k\to\infty} \mathbb{P}\{T_x^k - T_x^{k-1} < \infty, T_x^{k-1} - T_x^{k-2} < \infty, \dots, T_x^1 < \infty\}$$

$$\overset{!}{=} \lim_{k\to\infty} \mathbb{P}\{T_x^k - T_x^{k-1} < \infty\}^k = 1.$$

—×— contradiction

↑ this shows that (*) ⟺ recurrence

Now assume that $\boxed{\mathbb{P}\{T_x^1 < \infty\} = q < 1}$

Let's compute $\mathbb{P}\{R_x = m\}$.

- $\mathbb{P}\{R_x = 1\} = \mathbb{P}\{$chain never returns$\} = 1 - q$ (why?)
- $\mathbb{P}\{R_x = 2\} = \mathbb{P}\{$chain returns once and never again$\}$
  $= \mathbb{P}\{T_x^1 < \infty, T_x^2 - T_x^1 = \infty\} = q(1-q)$ (why?)

**Claim** $\mathbb{P}\{R_x = m\} = q^{m-1}(1-q)$ (Geometric! Makes sense?)

$$\Rightarrow \mathbb{E}(R_x) = \sum_{m=1}^{\infty} m\,\mathbb{P}\{R_x = m\} = \sum_{m=1}^{\infty} m\,(1-q)\,q^{m-1}$$

$$= \frac{1}{1-q} < \infty$$

$$\Rightarrow \boxed{\frac{1}{1-q} = \mathbb{E}(R_x) = \sum_{n=0}^{\infty} p_n(x,x) < \infty.}$$

y-product formula: $\boxed{\sum_{n=0}^{\infty} p_n(x,x) = \frac{1}{1-\mathbb{P}\{T_x^1 < \infty\}} = \mathbb{E}(R_x)}$

xamples

Simple symmetric RW in $\mathbb{Z}^d$: Determine whether are recurrent or transient

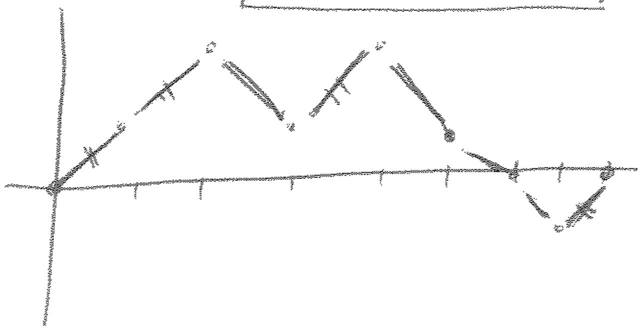
ase 1. $d=1$

$(x,x+1) = p(x,x-1) = \frac{1}{2}$. Note: This is a periodic chain!

Because it is irreducible, we need to focus on just one state, say 0 and assume that $X_0 = 0$.

We want to estimate the sum $\sum_{n=0}^{\infty} p_n(0,0)$.

First note that $p_n(0,0) = 0$ if $n$ is odd! So we need to evaluate $\boxed{\sum_{n=0}^{\infty} p_{2n}(0,0).}$



For the chain to return to 0, the walk took an equal number of left and right steps, and that number must be $\underline{n}$.

Thus $P_{2n}(0,0) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{n!\,n!} \left(\frac{1}{2}\right)^{2n}$.

We need to estimate $n!$ to manipulate the sum. Use Taylor's formula (proof later) $\boxed{n! \sim \sqrt{2\pi n}\; n^n e^{-n}}$.

So $P_{2n}(0,0) = \frac{(2n)!}{n!\,n!} \frac{1}{(2)^{2n}} \simeq \frac{\sqrt{4\pi n}\; \cancel{2^{2n}} n^{2n} e^{-2n}}{2\pi n\; n^n n^n e^{-n} e^{-n}} \cdot \frac{1}{\cancel{2^{2n}}}$

$= \frac{1}{\sqrt{\pi n}}$

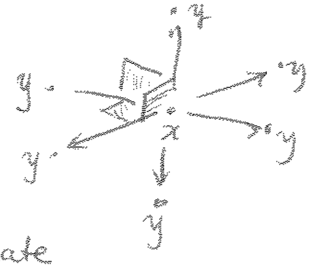
$\Rightarrow \sum_{n=1}^{\infty} P_{2n}(0,0) \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = +\infty$ (why?).

So <u>1-dimensional r.w. simple symm. is recurrent.</u>

# Lec. 17

## se 2  $d=3$

Transition probabilities: $p(x,y) = \begin{cases} \frac{1}{6} & \text{if } x \sim y \\ 0 & \text{o.w.} \end{cases}$



Again, for $n$ odd $P_n(0,0) = 0$, so we need to estimate $\sum_{n=0}^{\infty} P_{2n}(0,0)$.

As before, the walk must have taken equal # steps in each of the directions, say $i, j, k$, with $i+j+k=n$.

$$\Rightarrow P_{2n}(0,0) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \underbrace{\frac{(2n)!}{(i!\,j!\,k!)^2}}_{\text{why?}} \cdot \left(\frac{1}{6}\right)^{2n} = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{(2n)!}{n!\,n!} \cdot \frac{n!\,n!}{(i!\,j!\,k!)^2} \cdot \left(\frac{1}{6}\right)^{2n}$$

$$= \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i,j,k}^2 \cdot \left(\frac{1}{3}\right)^{2n} \qquad (*)$$

if $\boxed{n=3m}$, then $\binom{n}{i,j,k} \overset{(\dagger)}{\leq} \binom{3m}{m\ m\ m}$

$$\Rightarrow \quad (*) \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i,j,k}\binom{3m}{m\ m\ m}\frac{1}{3^n}\cdot\frac{1}{3^n}$$

$$= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\left(\frac{1}{3}\right)^{n}\binom{3m}{m\ m\ m}\underbrace{\sum \binom{n}{i,j,k}\frac{1}{3^n}}_{=1\ \text{(why?)}}$$

$(\dagger)$ $\frac{n!}{(i-1)!(j+1)!k!} = \frac{n!}{i!j!k!}\underbrace{\boxed{\frac{i}{j+1}}}_{\leq 1}$ if $i<j$

so highest value is when $i,j,k$ are equal. Hence the treatment of the case $n=3m$ separately.

$$\sim C\left(\frac{6}{n}\right)^{3/2}.$$

f $n=3m-1$, $3m-2$ $\longrightarrow$ $P_{6m}(0,0) \geq \frac{1}{6^2}P_{6m-2}(0,0)$

$P_{6m}(0,0) \geq \frac{1}{6^4}P_{6m-4}(0,0)$.

$$\Rightarrow \sum P_{2n}(0,0) < \tilde{C}\sum\left(\frac{6}{n}\right)^{3/2} < \infty. \quad \text{(why?)}$$

So 3-dimensional r.w. is transient!

**Theorem.**

Simple r.w. in $\mathbb{Z}^d$ is recurrent if $d=1$ or $2$ and transient if $d\geq 3$. ■

**Exercise 2.18. (Stirling's formula)**

Let $\{X_i\}_{1\leq i<\infty}$ an i.i.d. sequence of Poisson $(1)$ r.v.

$\left(\Rightarrow \quad \mathbb{P}\{X_i=k\} = \frac{e^{-1}}{k!}\right)$

Then $\sum_{i=1}^{n} X_i = Y_n \sim$ Poisson $(n)$ r.v.

Let $p(n,k) = \mathbb{P}\{Y_n = k\} = e^{-n} \dfrac{n^k}{k!}$ and let $a>0$.

By CLT

$$\mathbb{P}\left\{0 \le \frac{Y_n - n}{\sqrt{n}} \le a\right\} = P\left\{0 \le \frac{\sum X_i - n}{\sqrt{n}} \le a\right\}$$

$$\xrightarrow[n\to\infty]{} \int_0^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.$$

But.

$$\mathbb{P}\left\{0 \le \frac{Y_n - n}{\sqrt{n}} \le a\right\} = \sum_{k=n}^{n+a\sqrt{n}} \mathbb{P}\{Y_n = k\} = \sum_{k=n}^{n+a\sqrt{n}} p(n,k).$$

Hence,

$$\boxed{\lim_{n\to\infty} \sum_{k=n}^{n+a\sqrt{n}-1} p(n,k) = \int_0^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.} \quad (1).$$

Now,

$$p(n,k) = e^{-\cancel{a}n} \frac{n^k}{k!}$$ and assume $n \le k < n + a\sqrt{n}$.

Then $\dfrac{n!}{k!} \le n^{n-k}$ since we can do induction,

$$\Rightarrow \frac{n^k}{k!} \le \frac{n^n}{n!} \Rightarrow e^{-n}\frac{n^k}{k!} \le \frac{n^n}{n!}e^{-n} \Rightarrow \boxed{p(n,k) \le p(n,n)}. \quad (2)$$

Also for $n$ large

$$e^{-a^2} \underset{\overset{\uparrow}{\text{why?}}}{\lesssim} \left(1 - \frac{a}{\sqrt{n}+a}\right)^{a\sqrt{n}} \underset{\overset{\uparrow}{\text{why?}}}{\le} \left(\frac{n}{n+a\sqrt{n}}\right)^{k-n} \le \frac{n^{k-n}}{k^{k-n}} \le \frac{n^{k-n}\,n!}{k!}$$

$$\Rightarrow \boxed{e^{-a^2} p(n,n) \le p(n,k)} \quad (3)$$

From (1), (2), (3) $\Rightarrow$ $\boxed{p(n,n) \sim \dfrac{1}{\sqrt{2\pi n}}.}$ ∎

(Need to let $a \to 0$)

nother transience criterion. $\boxed{\text{Irreducible M.C.}}$ Lec. 18

Define $\alpha(x) = \mathbb{P}\{X_n = z \text{ for some } n \geq 0 \mid X_0 = x\}$.

$\alpha(z) = 1$

If chain is recurrent, then $\alpha(x) = 1 \quad \forall x$ !

f chain is transient, then there must be states with $\alpha(x) < 1$.

(makes sence?)

n fact, there should be states $x$ further and further away from $z$ with $\alpha(x)$ as small as we want.

$$\text{Also, } z \neq x, \ \alpha(x) = \mathbb{P}\{X_n = z \text{ for some } n \geq 0 \mid X_0 = x\}$$

$$= \mathbb{P}\{X_n = z \text{ for some } n \geq 1 \mid X_0 = x\}$$

$$= \sum_{y \in S} \mathbb{P}\{X_1 = y \mid X_0 = x\} \mathbb{P}\{X_n = z \text{ for some } n \underline{\underline{\geq 1}} \mid \underline{\underline{X_1}} = y\}$$

$$= \sum_{y \in S} p(x,y)\, \alpha(y).$$

or a <u>transient chain,</u> $\alpha(x)$ satisfies

(1) $0 \leq \alpha(x) \leq 1$

(2) $\alpha(z) = 1, \quad \inf\{\alpha(x) ; x \in S\} = 0$

(3) $\alpha(z) = \sum_{y \in S} p(x,y)\, \alpha(y), \quad x \neq z.$

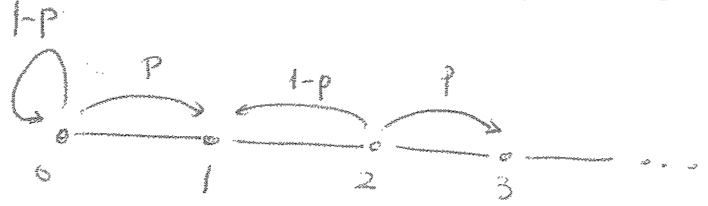
## Criterion.

$X_n$ recurrent $\iff$ no solution $\alpha(x)$ satisfying (1), (2), (3) exists.

$X_n$ transient $\iff$ ~~no~~ $\exists!$ unique solution to (1), (2), (3).

## Example.

Random walk with partially reflecting boundary.



$p(x,x-1) = 1-p \qquad x \geq 1.$

$p(x,x+1) = p \qquad x \geq 0.$

$p(0,0) = 1-p.$

Intuition?

Let $z=0$ and see if we can solve (1), (2), (3).

y (3) $\quad \alpha(x) = (1-p)\alpha(x-1) + p\,\alpha(x+1) \quad , x>0. \quad , \alpha(0)=1.$

$$\Longrightarrow \quad \alpha(x) = c_1 + c_2 \left(\frac{1-p}{p}\right)^x \qquad p \neq \frac{1}{2}$$

$$\alpha(x) = c_1 + c_2 \,.\, x \qquad p = \frac{1}{2}$$

le have that $\alpha(0)=1 \Rightarrow$

$$\alpha(x) = \begin{cases} (1-c_2) + c_2\left(\frac{1-p}{p}\right)^x & p \neq +\frac{1}{2} \\ 1 + c_2\, x & p = \frac{1}{2}. \end{cases}$$

If $c_2 = 0$ we cannot satisfy (2) [since $\alpha(x) = 1 \ \forall x$].

If $c_2 \neq 0$, $p = \frac{1}{2}$, $\alpha(x) \notin (0,1) \ \forall x.$

f $c_2 \neq 0$, $p < \frac{1}{2}$, $\alpha(x) \notin (0,1) \ \forall x.$ $\Big\} \rightarrow$ (1) is violated.

o, no matter what $c_2$ is, (1), (2), (3) cannot be simultaneously atisfied, so the chain is <u>recurrent</u> for $\underline{p \le \frac{1}{2}}$

or $p > \frac{1}{2}$, $\inf\{\alpha(x) : x \in S\} = 0 \iff \inf_{x \in S} \left\{ (1-c_2) + c_2 \left( \frac{1-p}{p} \right)^x \right\} = 0$

$$\iff \inf_{x \in S} \left\{ c_2 \left( \frac{1-p}{p} \right)^x \right\} - c_2 + 1 = 0$$

$$\iff -c_2 + 1 = 0 \quad \text{(why?)}, \quad 1 = c_2 \Rightarrow \boxed{\alpha(x) = \left( \frac{1-p}{p} \right)^x} \in (0,1)$$

lso, $\alpha(0) = 0$ as required.

Since a solution to (1), (2), (3) exist, the chain is transient!

## <u>ection 2.3</u> Positive & null Recurrence.

<u>Question 1</u>: $X_n$ is irreducible, aperiodic, MC, on an infinite state space $S$

Does this imply the existence of a limiting (invariant) probability?

A1: No! $X_n$ might be transient!

2: What if $X_n$ is recurrent?

2: Not always! ⤵

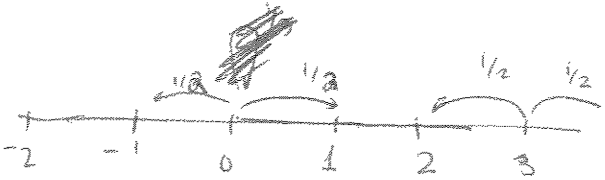
- A limiting probability distribution $\pi$ satisfies $\forall x, y \in S$
  $$\lim_{n \to \infty} P_n(y,x) = \pi(x)$$
- If $X_n$ is transient then $\lim_{n \to \infty} P_n(y,x) = 0$!

$\Longrightarrow$ No limiting $\pi(x)$ exists $\left( \sum_{x \in S} \pi(x) \overset{!}{=} 1 \right)$ should.

However, there's no guarantee if a chain is recurrent that

$$\lim_{n \to \infty} P_n(y, x) \neq 0.$$

For example, consider



~~This is irreducible, aperiodic, and~~ ~~$P_{2n+1}(0,0) > \frac{1}{3^{2n+1}} > 0$ (why?)~~

~~$P_{2n}(0,0) > \binom{2n}{n} \cdot \frac{1}{2^{n-2}} \cdot \frac{1}{3^2}$~~

~~Then show that $\sum_{n=0}^{\infty} P_n(0,0) = +\infty$, the chain is recurrent (why?)~~

~~But $\lim_{n \to \infty} P_n(0,0) \longrightarrow 0$ ( Show that~~

~~If you want to stay at 0 for $k$ times, then $n-k$ has to be even (why?)~~

~~Then $P_n(0,0) \overset{\text{why?}}{\leq} \sum_{k=0}^{n} \frac{1}{3^k} \binom{n-k}{\frac{n-k}{2}} \binom{n}{k} \frac{1}{2^{n-k}} \leq \binom{n}{\frac{n}{2}} \frac{1}{2^n} \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \binom{n}{k}$~~

Even, though it's 2-periodic, it is recurrent but $P_{2n}(0,0) \longrightarrow 0$.

Ve need to distinguish:

- If $X_n$ is an irreducible MC, aperiodic ~~with~~, recurrent with

$$P_n(x,y) \longrightarrow 0 \quad \Longrightarrow \quad \text{Null recurrent.}$$

O.W.

$$P_n(x,y) \longrightarrow \pi(x) > 0 \quad \Longrightarrow \quad \text{Positive recurrent.}$$

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Positive Recurrent Chains behave similarly to <u>finite</u> MC.

Facts: If $X_n$ irreducible, positive recurrent, aperiodic then

1) $$\lim_{n\to\infty} P_n(y,x) = \pi(x) > 0$$

2) $$\sum_{y\in S} \pi(y)\, p(y,x) = \pi(x) \quad (4)$$ is an invariant distribution.

3) The return time $T = \min\{n>0 \mid X_n = x\}$ has

$$\mathbb{E}\left[T \mid X_0 = x\right] = 1/\pi(x)$$

If $X_n$ is null recurrent, $\mathbb{E}(T_x^1) = +\infty$ but $T < \infty$ a.s.

The invariant distribution is unique.

If the chain is <u>not</u> positive recurrent, then no such $\pi$ (as in (4)) exist.

<u>Strategy</u>: To check positive recurrence, try and find a distribution satisfying (4). If you show that no such $\pi$ exist, you can't have positive recurrence.

<u>Example</u>: Random walk with partially reflecting boundary (again)

(4) gives

$$(1-p)\pi(x+1) + p\,\pi(x-1) = \pi(x) \qquad x > 0.$$

$$\pi(1)(1-p) + \pi(0)(1-p) = \pi(0) \implies \pi(0) = \pi(1)\left[\frac{1-p}{p}\right].$$

The solution to the recursion is

$$\pi(x) = c_1 + c_2 \left( \frac{p}{1-p} \right)^x , \quad p \neq \frac{1}{2} \qquad (a)$$

$$\pi(x) = c_1 + c_2 x \qquad p = \frac{1}{2} \qquad (b)$$

$$\pi(0) = \frac{1-p}{p} \pi(1) \qquad (*)$$

~~Plug (*) into (a) & (b)~~ , $\lim_{x \to \infty} \pi(x) = 0$ (why?)

$$\Rightarrow \quad \pi(x) = c_2 \left( \frac{p}{1-p} \right)^x \qquad p \neq \frac{1}{2}$$

$$\pi(x) = c_1 \qquad p = \frac{1}{2}.$$

Need: $\sum_{x \in \mathbb{N}_0} \pi(x) = 1$. $\Longrightarrow$ For $p = \frac{1}{2}$ we have null recurrence

For $p < \frac{1}{2}$, $1 = c_2 \sum_x \left( \frac{p}{1-p} \right)^x = c_2 \cdot \frac{1}{1 - \frac{p}{1-p}} = c_2 \frac{1-p}{1-2p}$.

$$\Rightarrow \quad c_2 = \frac{1-2p}{1-p}.$$

$$\Rightarrow \quad \boxed{\pi(x) = \left( \frac{1-2p}{1-p} \right) \left( \frac{p}{1-p} \right)^x} \qquad (*).$$

# Lec. 20

## Discrete time, Countable state space MC:

For the purposes of the diagram, assume that we have an irreducible MC on $\mathbb{N} \cup \{0\}$. Of course, the diagram is true in more generality.

Irreducible MC (so we only need to consider one state)

**Ways to show Transient**

1. $\sum_{n=0}^{\infty} p_n(i,i) < \infty$ for some state $i$

2. $\exists\, i$ s.t. $\mathbb{P}\{\text{ever returning to } i \mid X_0 = i\} < 1$

$\Leftrightarrow$

3. $\exists\, i$ s.t. $\mathbb{P}\{\text{Never returning to } i \mid X_0 = i\} > 0$

. $\exists$ a function $\alpha(x)$ s.t

$0 \le \alpha(j) \le 1$

$\alpha(i) = 1$, $\inf\{\alpha(j) : j \in S\} = 0$

$\alpha(j) = \sum_{k \in S} p(j,k)\, \alpha(k) \quad i \ne j \qquad (*)$

$(j) = \mathbb{P}\{X_n = i \text{ for some } n \mid X_0 = j\}$

**Ways to show Recurrence**

1. $\sum_{n=0}^{\infty} p_n(i,i) = +\infty$ for some state $i$

2. $\exists\, i$ s.t $\mathbb{P}\{\text{ever returning to } i \mid X_0 = i\} =$

$\Leftrightarrow$

3. $\exists\, i$ s.t $\mathbb{P}\{\text{Never returning to } i \mid X_0 = i\} =$

4. No such function $\alpha(x)$ exists

Positive Recurrent

a) Find a convenient state $i$ to compute the expected time of first return $\mathbb{E}[T^1_{ii}] = \sum_{n=0}^{\infty} n \cdot \mathbb{P}\{T^1_{ii} = n\} < \infty$

b)

Next time

Null recurrent

a) Find a conv. state $i$ to compute expected time of first return $\mathbb{E}[T^1_{ii}] = \infty$

b)

Next time

## Positive Recurrence

b) Find a function $\pi(x)$ s.t.

$$\left\{ \begin{array}{l} 0 \leq \pi(x) \leq 1 \\ \sum_{x \in S} \pi(x) = 1 \\ \pi(x) = \sum_{y \in S} \pi(y)\, p(y,x) \quad \forall x \in S \end{array} \right\}$$

$\pi$ is the invariant distribution, and

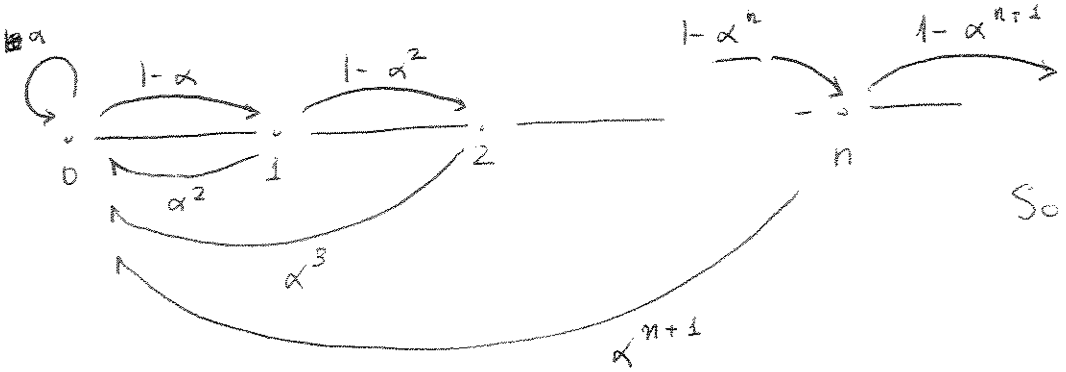
$$\pi(x) = \frac{1}{\mathbb{E}(T^1_{xx})} > 0$$

## Null Recurrence

b) Such function $\pi$ cannot exist.

## Examples

Let $0 < \alpha < 1$ and consider the MC (Give students ~7 minutes to work alone) then do on board



So $p(n,n+1) = 1-\alpha^{n+1}$

$p(n,0) = \alpha^{n+1}$

1. Show ~~the~~ explain/check this is aperiodic.
2. Check whether is recurrent or transient, using (3).

- Compute $\mathbb{P}\{\text{we never return to } 0 \mid X_0 = 0\} =$

$$= (1-\alpha)(1-\alpha^2)(1-\alpha^3)\cdots(1-\alpha^n)\cdots = \prod_{n=1}^{\infty}(1-\alpha^n)$$

We say the infinite product converges iff $\lim_{n\to\infty} \prod_{i=1}^{n} (1-\alpha^i)$ exists and it not equal to 0

A useful criterion for convergence of infinite products of this form:

$$\prod_{i=1}^{\infty} (1 \pm u_i) \text{ converges iff } \sum_{i=1}^{\infty} u_i < \infty$$

(Try and prove it by yourselves)

In this case, $u_i = \alpha^i$

$\Rightarrow$ Check $\sum_{i=1}^{\infty} \alpha^i = \frac{\alpha}{1-\alpha} < +\infty$. Hence the product converges and is positive. Thus,

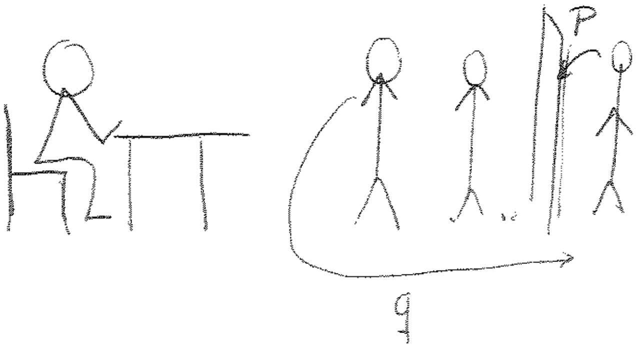
$\mathbb{P}\{\text{never return to } 0 \mid X_0 = 0\} > 0$, so the chain is transient. ▨

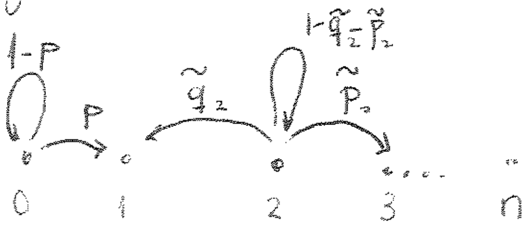
## Example 2: Single server discrete time queue.

Let $X_n$ be the number of customers waiting to get serviced in a single queue. Let $0<p,q<1$

The probability of a customer arrival in a time window is $p$

The probability of a customer departure in a time window is $q$ as long as there is a customer to get serviced!





$\tilde{p}_x = p(1-q)$ why? $= p(x,x+1)$ $x>0$ $p(0,0) = 1-p = \tilde{p}_0$

$\tilde{q}_x = q(1-p)$ why? $= p(x,x-1)$ $x \geq 1$ $p(0,1) = p = \tilde{q}_0$

Assumption A customer can't served in the same time frame as he arrived

# Recurrence / Transience for single server queue

We use (4).

Let $\alpha(n) = \mathbb{P}\{\text{we reach } 0 \mid X_0 = n\} = \alpha_n$

Then (either by one-step analysis) or by $(*)$, we have

$$\alpha_n = \tilde{p}_n \alpha_{n+1} + \tilde{q}_n \alpha_{n-1} + (1-\tilde{p}_n-\tilde{q}_n)\alpha_n \,, \quad n>0 \iff$$

$$(\tilde{p}_n+\tilde{q}_n)\alpha_n = \tilde{p}_n \alpha_{n+1} + \tilde{q}_n \alpha_{n-1} \qquad n \geq 1 \qquad \overset{\text{ALGEBRA}}{\iff}$$

$$\alpha_n - \alpha_{n+1} = \frac{\tilde{q}_n}{\tilde{p}_n}\left[\alpha_{n-1} - \alpha_n\right] \qquad n \geq 1.$$

Iterating yields

$$\alpha_n - \alpha_{n+1} = \frac{\tilde{q}_1 \tilde{q}_2 \cdots \tilde{q}_n}{\tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_n}\left[\alpha_0 - \alpha_1\right] \underset{\text{why?}}{\overset{!}{=}} \frac{\tilde{q}_1 \cdots \tilde{q}_n}{\tilde{p}_1 \cdots \tilde{p}_n}\left[1-\alpha_1\right].$$

Then,

$$\alpha_{n+1} = \left[\alpha_{n+1} - \alpha_0\right] + \alpha_0 = \sum_{k=0}^{n}(\alpha_{k+1} - \alpha_k) + 1$$

$$= 1 - \left[1-\alpha_1\right]\sum_{k=0}^{n} \frac{\tilde{q}_1 \cdots \tilde{q}_k}{\tilde{p}_1 \cdots \tilde{p}_k}$$

$$= 1 - C \sum_{k=0}^{n} \left(\frac{\tilde{q}}{\tilde{p}}\right)^k \qquad (**)$$

If the sum $(**)$ diverges (as $n \to \infty$) then the first two equations in (4) cannot be satisfied, so the chain is [redacted] recurrent. The sum converges only if $\tilde{p} > \tilde{q}$ $\iff$

$\Longleftrightarrow$

$$p(1-q) > q(1-p) \quad \Longleftrightarrow \quad p > q$$

- **Lemma** 🙂 . If $p > q$ the chain is transient and the server should either be fired or get a medal
- If $p \leq q$ the chain is recurrent

**Question**: When do you think we have null recurrence?

**Invariant distributions.**

We use the set of equations in (b). Then

$$\pi(x) = \pi(x-1)\,\tilde{p}_{x-1} + \pi(x)\left(1-\tilde{p}_x-\tilde{q}_x\right) + \pi(x+1)\,\tilde{q}_{x+1} \qquad x \geqslant 1 \quad (\dagger)$$

$$\pi(0) = \pi(0)\left(1-\tilde{p}_0\right) + \pi(1)\,\tilde{q}_1 \qquad (\dagger\dagger)$$

$$\Leftrightarrow \quad \tilde{q}_1\,\pi(1) - \tilde{p}_0\,\pi(0) = 0$$

From $(\dagger)$ & algebra,

$$\tilde{q}_{x+1}\,\pi(x+1) - \tilde{p}_x\,\pi(x) = \tilde{q}_x\,\pi(x) - \tilde{p}_{x-1}\,\pi(x-1)$$

$$= \dots$$

$$= \tilde{q}_1\,\pi(1) - \tilde{p}_0\,\pi(0)$$

$$= 0$$

$$\Longleftrightarrow \quad \boxed{\pi(x+1) = \frac{\tilde{p}_x}{\tilde{q}_{x+1}}\,\pi(x)} \qquad \forall x \geqslant 0$$

$$= \frac{\tilde{p}_0\,\tilde{p}_1 \cdots \tilde{p}_{x-1}}{\tilde{q}_1\,\tilde{q}_2 \quad \tilde{q}_x}\,\pi(0) = \frac{\tilde{p}_0}{\tilde{p}}\left(\frac{\tilde{p}}{\tilde{q}}\right)^{x-1} \cdot \pi(0)$$

$$= \left(\frac{\tilde{p}}{\tilde{q}}\right)^{x} \cdot \pi(0)$$

Then , we want .

$$\sum_{x=0}^{\infty} \pi(x) = 1 \Rightarrow \sum_{x=1}^{\infty} \overset{1}{\cancel{\frac{\tilde{p}}{\tilde{p}}}} \left(\frac{\tilde{p}}{\tilde{q}}\right)^{x} \pi(0) + \pi(0) = 1$$

$$\Rightarrow \pi(0)\left[1 + \sum_{x=1}^{\infty} \left(\frac{\tilde{p}}{\tilde{q}}\right)^{x}\right] = 1$$

$$\Rightarrow \pi(0) = \frac{1}{1 + \sum_{x=1}^{\infty} \left(\frac{\tilde{p}}{\tilde{q}}\right)^{x-1}}$$

$> 0$ iff $\tilde{p} < \tilde{q}$

$\Rightarrow \underline{\tilde{p} = \tilde{q} \text{ Null}}$ recurrence

$$= \frac{1}{1 + \frac{\tilde{p}/\tilde{q}}{1 - \tilde{p}/\tilde{q}}} = 1 - \tilde{p}/\tilde{q}$$

$$\Rightarrow \pi(x) = \pi(0)\left(\frac{\tilde{p}}{\tilde{q}}\right)^{x} = \left(1 - \tilde{p}/\tilde{q}\right)\left(\frac{\tilde{p}}{\tilde{q}}\right)^{x} \quad \blacksquare$$

Lec. 21&22 were on random walk in random environment

will post later.

# Lec. 23

Branching process

. A population of individuals reproduces at each time step $n$.

$X_n$ = # of individuals at time $n$

Each individual produces a random number of offsprings, and after producing them, the individual dies! (A horrible, horrible death)

Assumptions on reproduction process

1. Each individual produces offsprings with the same probability distribution: $\exists$ $p_0, p_1, \dots, p_k$, $\sum_{i=0}^{\infty} p_i = 1$, so that the individual has $k$ children with prob. $p_k$.

2. Individuals reproduce independently!

Notation: The offspring distribution.

Individual $i$ produces $Z_i^n$ offsprings at time $n$

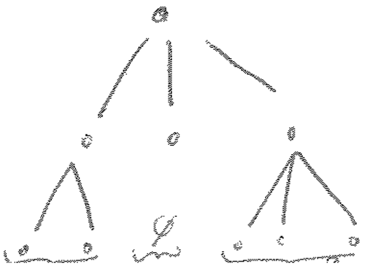
with $\mathbb{P}\{Z_i^n = j\} = p_j$ $\quad 0 \le j < \infty$.

Why is it called a branching process? Example

$X_0 = 1$

$X_1 = 3 = Z_1^1$

$X_2 = Z_1^2 + Z_2^2 + Z_3^2$



. In general $X_n = \sum_{i=1}^{K} Z_i^n$ where

$K = X_{n-1}$ (This should make sense!)

Transition probabilities ...

Are extremely hard to compute! For example:

$$p(k,j) = \mathbb{P}\{X_{n+1} = j \mid X_n = k\} = \mathbb{P}\{Z_1^{n+1} + Z_2^{n+1} + \cdots + Z_k^{n+1} = j\}$$

with $Z_i^{n+1}$ i.i.d.

However, we can compute $p(0,0)$ ! ( $p(0,0) = 1$ ! why?)

A natural set of questions!

1: What is the expected number of population at level $n$?

2: ~~How can~~ Do we know if the population will go extinct?

3: Can we say when the population will die?

1. Let $\mu = \mathbb{E}(Z_i^n)$ = mean number of offsprings of an individual

. By definition $\mu = \sum_{i=0}^{\infty} i p_i$

Then,
$$\mathbb{E}[X_{n+1} \mid X_n = k] = \mathbb{E}(Z_1^{n+1} + Z_2^{n+1} \ldots Z_k^{n+1})$$
$$= k\mu$$

Then

$$\mathbb{E}(X_n) \overset{!}{=} \sum_{k=0}^{\infty} \mathbb{P}\{X_{n-1}=k\}\, \mathbb{E}(X_n \mid X_{n-1}=k) \qquad \text{(Definition)}$$

$$= \sum_{k=0}^{\infty} k\mu\, \mathbb{P}\{X_{n-1}=k\} = \mu \sum_{k=0}^{\infty} k \cdot \mathbb{P}\{X_{n-1}=k\} \qquad (\dagger)$$

$$\overset{!}{=} \mu \cdot \mathbb{E}[X_{n-1}] \qquad \text{( recursion formula)}$$

Iterate the procedure $\Rightarrow$ $\mathbb{E}[X_n] = \mu^n \, \mathbb{E}[X_0]$. $(*)$

2] ~~Example~~

. Assume $\mu < 1$. Then $(*)$ says $\mathbb{E}[X_n] = c \cdot \mu^n \longrightarrow 0$ $(n \to \infty$.

Now observe that since by $(\dagger)$

$$\mathbb{E}[X_n] = \sum_{k=0}^{\infty} k\, \mathbb{P}\{X_n = k\} \overset{!}{\geq} \sum_{k=1}^{\infty} \mathbb{P}\{X_n = k\} = \mathbb{P}\{X_n \geq 1\}$$

So $\mathbb{P}\{X_n \geq 1\} \xrightarrow[n\to\infty]{} 0$. Thus $\mathbb{P}\{X_n = 0\} \geq 1 - c\mu^n$

$\longrightarrow 1$ exponentially fast.

Fact

If $\mu < 1$,

$\mathbb{P}\{$ population dies eventually out $\} = 1$

**Q4:** What about when $\mu = 1$, $\mu > 1$? Intuition?

A useful tool: Probability generating function.

# Lec. 24

For a r.v. $X$, taking values in $\{0, 1, 2, \dots\}$ the <u>probability generating function</u> (or generating function)

$$\boxed{\phi_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k \cdot \mathbb{P}\{X=k\}}$$

Assume that $0 \le s \le 1$ so that limit always exists. [redacted]

<u>Example</u>:

Let $X$ be a Poisson$(\lambda)$ $\Longrightarrow$ $\mathbb{P}\{X=k\} = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$

Find $\phi_X(s)$.

<u>nswer:</u> $\phi_X(s) = \mathbb{E}(s^X) = \displaystyle\sum_{k=0}^{\infty} s^k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}.$$

<u>Properties of $\phi_X(s)$</u>

$$P(X=0) \le \phi(s) \le P(X=0) + s \sum_{k=1}^{\infty} P(X=k) \le P(X=0) + s$$

$$\text{So } \phi(s) \xrightarrow[s\to 0]{} P(X=0)$$

$\phi_X(0) = \mathbb{P}\{X=0\}$ (why?)

$\phi_X(1) = 1$ (why?)

$\phi_X(s)$ is increasing in $s$ (why?)

unless $P(X=0)=1$, in which case $\phi(s)=1$ $\forall s$

$$\hookrightarrow \phi_X'(s) = \sum_{k=1}^{\infty} k s^{k-1} \mathbb{P}\{X=k\} > 0$$

$$\phi''(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} \mathbb{P}\{X=k\}$$

Now observe

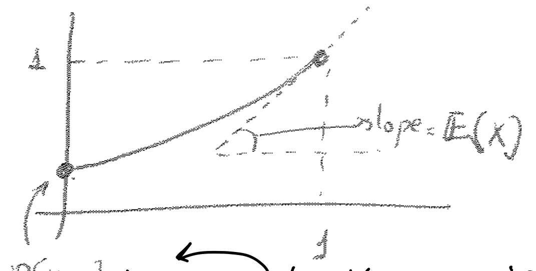
$$\phi_X'(1) = \sum_{k=1}^{\infty} k\,\mathbb{P}\{X=k\} = \mathbb{E}[X]$$

Note:
$P(X \geq 2) = 0 \longrightarrow X_n \in \{X_{n-1}, X_{n-1}+1\}$
Silly! $P(X=0) > 0 \Rightarrow$
$X_n$ eventually $= 0$
$P(X=1) = 1 \Rightarrow X_n = X_0 + n$

So can assume

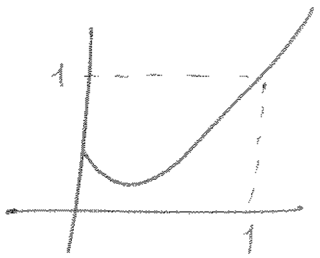
$$\phi_X''(s) > 0 \iff \mathbb{P}\{X \geq 2\} > 0 \Rightarrow \text{convex}$$

General shape



$\mathbb{P}\{X=0\} > 0$ ← o/w $X_n \to \infty$ is clear

Questions



Can it look like this?

Not really! why?

ndependence: Show that $\varphi_{X_1+X_2+\cdots+X_m}(s) = \varphi_{X_1}(s) \cdots \varphi_{X_m}(s)$ $X_i$ ind

Why is $\phi_X(s)$ useful?

For branching processes: Assumptions: $p_0 > 0$ ; $p_0 + p_1 < 1$

$P(X=0) > 0$

$P(X \geq 2) > 0$

(why are these assumptions reasonable?)

Define: $\alpha_n$ [redacted] $= \mathbb{P}\{X_n = 0 \mid X_0 = 1\} = \mathbb{P}\{\text{we are dead by time } n\}$

$\alpha$ [redacted] $= \mathbb{P}\{\exists\, n \text{ s.t. } X_n = 0 \mid X_0 = 1\}$

$\alpha$ [redacted] $= \lim_{n\to\infty}$ [redacted] $\alpha_n$ [redacted]

Now, if the population has $k$ individuals at a given time the only way to die out in the future is if <u>all</u> branches die out.

Then, ~~$\alpha(k) = \alpha(1)^k$~~ because branches behave independently

Prob. of extinction is $\alpha^k$

~~For all analysis purposes, it suffices to determine $\alpha(1) = \alpha$~~

~~Definition $\alpha = \alpha(1)$ is called the extinction probability~~

Silly computation (But extremely important)

$a = \mathbb{P}\{$ population eventually dies out $| X_0 = 1\}$

$= \sum_{k=0}^{\infty} \mathbb{P}\{X_1 = k \mid X_0 = 1\} . \mathbb{P}\{$ population dies out eventually $| X_1 = k\}$

~~$= \sum_{k=0}^{\infty}$ [illegible]~~

$= \sum_{k=0}^{\infty} p_k \alpha^k = \phi_Z(\alpha)$

Thus, the extinction prob is a fixed point of $\phi_Z(s)$. i.e.

$$\boxed{\alpha = \phi_Z(\alpha)}$$

Note: $1 = \phi_Z(1)$ for all $Z$ r.v. offspring distributions, but there might be other solutions

Assume $X_0 = 1$. Let $\phi_Z(s)$ be the offspring distribution prob. g. f

Define $g_n(s)$ to be the generating function of $X_n$.

- $g_0(s) = \mathbb{E}(s^{X_0}) = s . \mathbb{P}\{X_0 = 1\} = s$
- $g_1(s) = \mathbb{E}(s^{X_1}) = \sum_{k=0}^{\infty} s^k . \mathbb{P}\{X_1 = k\} = \sum_{k=0}^{\infty} s^k . \mathbb{P}\{Z = k\} = \phi_Z(s)$

**Proposition.**

$$g_n(s) = \underbrace{\phi_Z \circ \phi_Z \circ \cdots \circ \phi_Z}_{n\text{ -times}}(s) = \phi_Z^{(n)}(s)$$

composition !

**Proof.**

By induction, with n=1,0 as base cases.

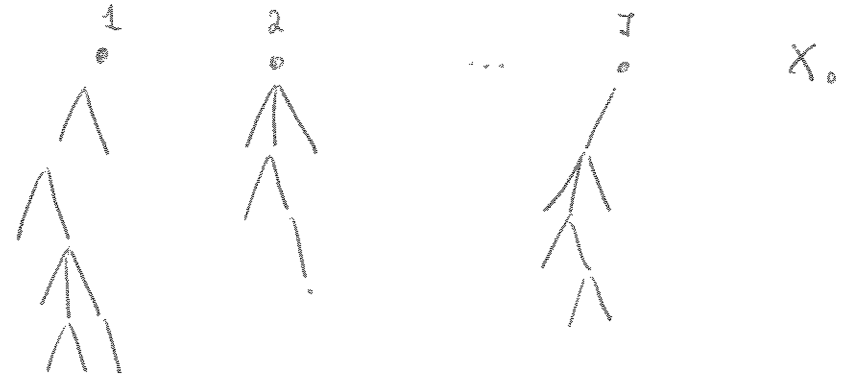
$$g_n(s) = \sum_{k=0}^{\infty} P\{X_n = k\}\, s^k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P\{X_1 = j\} . P\{X_n = k \mid X_1 = j\}\, s^k$$

$$\overset{!}{=} \sum_{j=0}^{\infty} p_j \sum_{k=0}^{\infty} s^k\, P\{X_{n-1} = k \mid X_0 = j\}$$

$$= \sum_{j=0}^{\infty} p_j \underbrace{\left( \sum_{k=0}^{\infty} s^k\, P\{X_{n-1} = k \mid X_0 = j\} \right)}_{\underbrace{[g_{n-1}(s)]^j}} \qquad (\dagger)$$

→ Proof of this !



$$X_{n-1} \qquad \underbrace{X^1_{n-1} \qquad X^2_{n-1} \qquad X^J_{n-1}}_{\text{I.I.D, (why?}}$$

$$\Rightarrow P\{X_{n-1} = k \mid X_0 = j\} = P\left\{ \sum_{i=1}^{J} X^i_{n-1} = k \,\middle|\, X^i_0 = 1 \right\}$$

Therefore $\sum_{k=0}^{\infty} s^k \, P\left\{ \sum_{i=1}^{j} X_{n-1}^{i} = k \right\}$

$$= g_{X_{n-1}^{1} + X_{n-2}^{2} + \cdots + X_{n-2}^{j}}(s)$$

$$= g_{X_{n-1}^{1}}(s)\, g_{X_{n-1}^{2}}(s) \cdots g_{X_{n-1}^{j}}(s)$$

$$= \left( g_{n-1}(s) \right)^{j}$$

Then $(\dagger) = \sum_{j=0}^{\infty} p_j \left(g_{n-1}(s)\right)^j \underset{\text{why?}}{\overset{!}{=}} \sum_{j=0}^{\infty} p_j \left[\phi_Z^{(n-1)}(s)\right]^j = \phi_Z\left(\phi_Z^{(n-1)}(s)\right)$

$$= \phi_Z^{(n)}(s). \quad ■$$

orollary (This answers $\mathbb{P}\{X_n = 0 \mid X_0 = 1\}$)

$$a_n = \mathbb{P}\{X_n = 0 \mid X_0 = 1\} \overset{!}{=} g_n(0) = \phi^{(n)}(0)$$

b/c $\phi_X(0) = P(X=0)$

Theorem

Lec. 25

he extinction probability $\alpha$ is the smallest possible solution

f $\alpha = \phi(\alpha)$

Proof

Let $\hat{a}$ be the smallest solution such that $\hat{a} = \phi(\hat{a})$

Let $a_n = P\{X_n = 0\}$. We'll show $a_n \le \hat{a}$

b/c $\phi(0) = p_0 \neq 0$ $\quad$ $\phi$ increasing (!)

Then $a_0 = 0 < \hat{a}$

$a_1 = \mathbb{P}\{X_1 = 0\} = \phi(0) \le \phi(\hat{a}) = \hat{a}$

Induction again: Assume true for $n-1$

$$a_n = \mathbb{P}\{X_n = 0\} = \phi^n(0) = \phi(\phi^{n-1}(0)) \leq \phi(\hat{a}) = \hat{a}$$

So $\overset{\lim_{n\to\infty}\alpha_n}{\downarrow}$ $\alpha = \mathbb{P}\{$ population eventually dies out $\}$ $\leq \hat{a}$

and $\overset{\text{we already know}}{\alpha = \phi(\alpha)}$. But $\hat{\alpha}$ is $\underbrace{}_{\text{by its definition!}}$ the smallest number such that $\hat{\alpha} = \phi(\hat{\alpha})$. So we also have $\alpha \geq \hat{\alpha}$.

Thus: $\boxed{\alpha = \hat{\alpha}}$

**Question**: When is $\alpha < 1$ (so we can survive?)

Theorem

if $\mu = \mathbb{E}[Z] \leq 1$ and $p_0 > 0$, the extinction probability $\alpha = 1$

if $\mu > 1$ ~~and~~ then $\alpha < 1$ and is the unique root (not equal 1 with $\phi(\alpha) = \alpha$.

Proof.

If $\mu < 1$ done.

If $\mu = 1$, $\phi_Z'(1) = 1$ and $\phi_Z'' > 0$. Since $\phi_Z'' > 0 \Rightarrow$

$\Rightarrow \phi_Z' \nearrow \Rightarrow \phi_Z'(s) < 1 \quad \forall s \in (0,1)$.

Then

$$1 - \phi(s) = \int_s^1 \phi'(s)\,ds < \int_s^1 1\,ds = 1 - s \Rightarrow \boxed{s < \phi(s)} \quad \forall s \in (0,1)$$

if $\mu > 1 \Rightarrow \phi_Z'(1) > 1 \Rightarrow \exists s_0 : \phi(s_0) < s_0$ (why?)

$f(s) = \phi(s) - s$ : $f'(s) = \phi'(s) - 1$, so $f'(1) > 0$

so near $s=1$ $f \nearrow$

but $f(1) = 0$.

so a little before $s=1$, $f$ must be $< 0$.

But $\phi(0) > 0$ so the function $\phi(s)$ crosses the diagonal for some $\alpha \in (0, s_0)$. ∎

D. $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$ $\mu = 1\frac{1}{4} > 1$ $\alpha < 1$

$p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$ $p_2 = \frac{1}{4}$ $\mu = \frac{3}{4} < 1$ $\alpha = 1$

$p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$ $p_2 = \frac{1}{4}$ $\mu = 1$ $\alpha = 1$

In the first example: $\alpha$ solves $\phi(\alpha) = \alpha$

$$\phi(\alpha) = \alpha^0 p_0 + \alpha^1 p_1 + \alpha^2 p_2 = \frac{1}{4} + \frac{\alpha}{4} + \frac{\alpha^2}{2}$$

So: $\alpha = \frac{1}{4} + \frac{\alpha}{4} + \frac{\alpha^2}{2} \Rightarrow 2\alpha^2 - 3\alpha + 1 = 0$

$$\Rightarrow (\alpha - 1)(2\alpha - 1) = 0$$

so $\alpha = 1$ or $\alpha = \frac{1}{2}$

$\alpha$ is the smallest solution.

So $\boxed{\alpha = \frac{1}{2}}$ 50% chance of survival.