-hapter 1 - Definitions & Examples Lec.1

- Definition: Stochastic process

A family of random variables $\{X_i\}_{i \in I}$ where i is an index set, usually denoting time (so $I = \mathbb{N} \cup \{0\}$, or $\mathbb{R}_+$)

RST PART OF THE COURSE – Discrete time

- A s.p. is now $\{X_n\}_{n \in \mathbb{N}}$

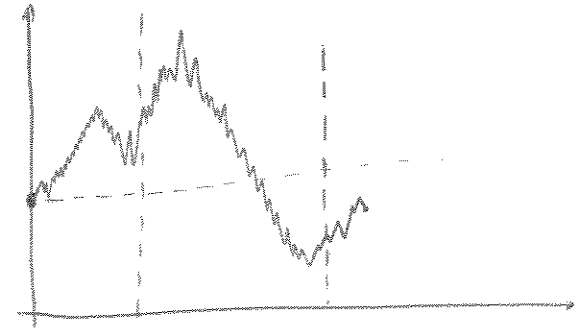
- each $n \in \mathbb{N}$, X_n is a random variable, i.e. a function from probability space Ω into a state space S

S can be: Finite / Infinite $<$ Countable / Uncountable

xamples of Stochastic Process

- ??? Random Walk, ~~Sequence~~ SUM of i.i.ds.

Example (Closing Value of Stock).



- Stock AWESOME.COM.CY has a value after the end of each day, when the market closes.

- What are the questions of interest?

- Long term behavior?
- Is day-to-day information enough?
- Does information we posses matters?

Answer to last question? YES! History changes probabilities!

Conditional probabity → We use information we already have

- Event B, $\mathbb{P}\{B\} > 0$ (necessary)
- We have knowledge of event B - we know it happened.
- Q: What is the prob. of something else happening?

Then

$$\mathbb{P}\{A|B\} = \frac{P(A\cap B)}{P(B)} := \frac{\mathbb{P}\{AB\}}{P\{B\}} = \frac{\mathbb{P}\{A,B\}}{P\{B\}}$$

· Note: $\boxed{P(A\cap B) = P(B)P(A|B)}$

. Disjoint events: (= Mutually exclusive)

(i.e. Sunny vs Rainy) $P\{A\cap B\} = 0 \Rightarrow \mathbb{P}\{A|B\} = 0$

Independent events: $\mathbb{P}\{A\cap B\} = \mathbb{P}\{A\}\mathbb{P}\{B\}$

(information about ~~the~~ **B** is uselen in terms of info about **A**)

$\Longrightarrow \quad \mathbb{P}\{A|B\} = \mathbb{P}\{A\}$.

Lecture 23/08 End

The three tools of conditioning

) . Multiplication Rule:

$$P(A_1 \dots A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1)\,\mathbb{P}(A_3|A_2\cap A_1)\dots\mathbb{P}(A_n|A_{n-1}\cap\dots\cap A_1)$$

(This is part of the first hw)

Example.

There are N keys in a basket and only one opens a door. What is the probability that the door will open at the n-th trial? $(1 \le n \le N)$

Define $A_i = \{i\text{-th key did not open door}\}$.

We want $\mathbb{P}\{A_1 .. A_{n-1} A_n^c\} =$ (why?) (†)

Then, by the multiplication rule:

$$(\dagger) = \frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdot\frac{N-3}{N-2}\cdots\frac{N-n+1}{N-n+2}\cdot\frac{1}{N-n+1} = \frac{1}{N} \quad \text{(Wow!)}$$

ty? Use permutations of the N keys !

$\exists$ (N-1)! that place the correct key at the n-th position

N! all perms.

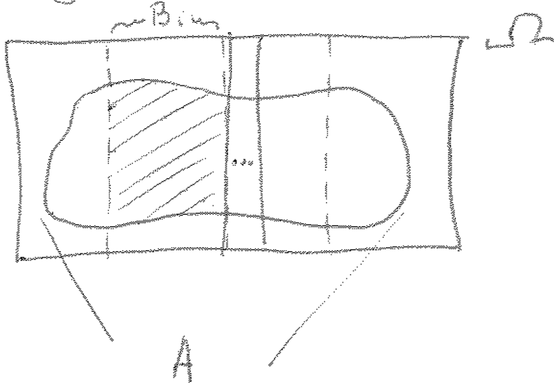
$$\Longrightarrow \quad \mathbb{P}\{A_1 \dots A_{n-1} A_n^c\} = \frac{(N-1)!}{N!} = \frac{1}{N} \quad !$$

3). Law of total probability.

t $\{B_i\}_{i \in \mathbb{N}}$, $\mathbb{P}\{B_i\} > 0 \quad \forall i$ (finite or countably many)

nd so that $\mathbb{P}\{B_i \cap B_j\} = 0$ if $i \neq j$ and $\sum_{i=1}^{\infty} \mathbb{P}\{B_i\} = 1$

They partition Ω!)



$$\boxed{\mathbb{P}\{A\} = \sum_{i=1}^{\infty} \mathbb{P}\{A \cap B_i\} \overset{!}{\underset{\text{why?}}{=}} \sum_{i=1}^{\infty} \mathbb{P}\{A \mid B_i\}\,\mathbb{P}\{B_i\}.}$$

c) Bayes Formula (aposteriori probability)

- We know an event happened. What is the prob. that something else happened before it ??

Baby version!

$$\mathbb{P}\{A \mid B\} = \frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{B\}} = \frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{A\}} \cdot \frac{\mathbb{P}\{A\}}{\mathbb{P}\{B\}} = \mathbb{P}\{B \mid A\} \cdot \frac{\mathbb{P}\{A\}}{\mathbb{P}\{B\}}$$

ardcore version

$$\mathbb{P}\{A \mid B\} = \frac{\mathbb{P}\{B \mid A\} \cdot \mathbb{P}\{A\}}{\underbrace{\sum_j \mathbb{P}\{B \mid A_j\} \cdot \mathbb{P}\{A_j\}}}$$

What is this?
What properties satisfy the A_i ?

Ex. Two coins. $P(\text{coin}1) = \frac{2}{3}$.
coin 1: $P(H) = \frac{1}{3}$, coin2 : $P(H) = \frac{3}{4}$
$P(\text{coin}2 \mid H) = ?$

Chapter 1 - Finite Markov Chains

Definitions & Examples

- Discrete time stochastic process, $\{X_n\}_{\mathbb{N}\cup\{0\}}$ Index "n" denotes time, (or time step)
- State space S is finite for this chapter!

he most natural thing to evaluate is the probabilities of the process:

$$\boxed{(\dagger) \quad \mathbb{P}\{X_0 = i_0 , X_1 = i_1 , \dots , X_n = i_n\}}$$

$'n \in \mathbb{N} , \forall (i_0, \dots , i_n) \in S^{n+1}$

obability $(\dagger)$ is an intersection of events, so, by the multiplication rule

$$(\dagger) = \underbrace{\mathbb{P}\{X_0 = i_0\}}_{\text{Initial distribution}} \cdot \underbrace{\mathbb{P}\{X_1 = i_1 \mid X_0 = i_0\} \dots \mathbb{P}\{X_n = i_n \mid X_{n-1} = i_{n-1} , \dots , X_0 = i_0\}}_{\text{Transition probabilities}} \quad (**)$$

n we simplify $(**)$? NOT ALWAYS

the process is MARKOV then it simplifies greatly:

- Markov Property (Restarting!)
  - In normal words: In order to make predictions for the future of the system, all you need to know is it's present state (not it's past history).
  - In mathy words: Conditional on the present, past and future are independent.
  - In mathy symbols: $\boxed{\mathbb{P}\{X_n = i_n \mid X_0 = i_0, \dots , X_{n-1} = i_{n-1}\} = \mathbb{P}\{X_n = i_n \mid X_{n-1} = i_{n-1}\}}$

Definitions

A markov chain (MC) is a stochastic process that satisfy the Markov Property.

A time-homogeneous MC satisfies $\mathbb{P}\{X_n = \underline{i_n} \mid X_0 = i_0, \ldots, X_{n-1} = \underline{i_{n-1}}\} = p(\underline{i_{n-1}}, \underline{i_n})$

for some function $p: S \times S \longrightarrow [0,1]$ that does not depend on n.

This implies $\boxed{\mathbb{P}\{X_{n+1} = j \mid X_n = j\} = p(i,j)}$ $\forall n \geq 0$.

The initial distribution ϕ gives the starting positions:

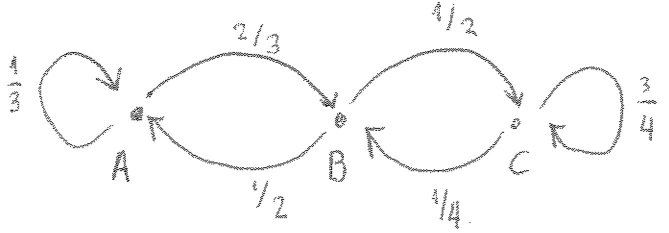
$$\boxed{\mathbb{P}\{X_0 = i\} = \phi(i)}$$

w (**) can be simplified as:

$$\mathbb{P}\{X_0 = i_0, X_1 = i_1, \ldots X_n = i_n\} = \phi(i_0)\, p(i_0, i_1)\, p(i_1, i_2) \cdots p(i_{n-1}, i_n)$$

zualization

Three state MC:



- Directed graph
- Is it a MC (Yes, why?)
- $p(B,C) = \frac{1}{2}$.

How to organise? From each state you go to a different state.

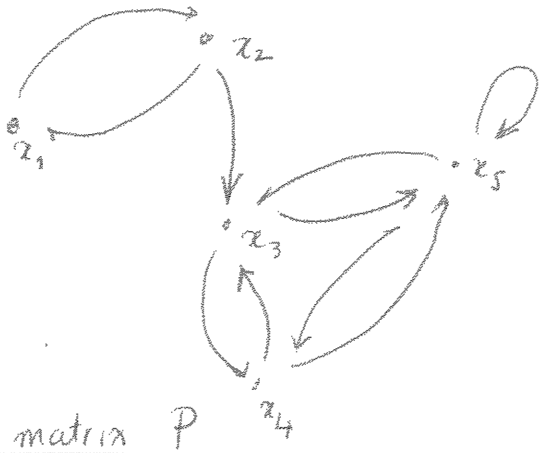
$\Rightarrow$ Two coordinates!

$$\begin{array}{c|ccc} & A & B & C \\ A & 1/3 & 2/3 & 0 \\ B & 1/2 & 0 & 1/2 \\ C & 0 & 1/4 & 3/4 \end{array} = P$$

- This is called a transition matrix (i.e. matrix of transition probabilities)
- It's a STOCHASTIC MATRIX. The rows sum up to 1 (why?).

In general, any directed graph, where the sum of outbounds probabili
sum to 1, can be turned into a MC.

(Why?) Because an MC is uniquely / completely defined by the transition probabilities, which in turn can be defined by the transition matrix.



$$P = \begin{bmatrix} p(x_1,x_1) & \cdots & p(x_1,x_5) \\ \vdots & p(x_i,x_j) & \\ p(x_5,x_1) & & p(x_5,x_5) \end{bmatrix}$$

tochastic matrix P

Entries $P_{ij} = p(i,j)$, $0 \leq P_{ij} \leq 1$ (why?)

Stochastic ~~Matrix~~ $\sum_{j=1}^{N} P_{i,j} = 1$ (row sum) (why?)

(why?)

Anything satisfying 1&2 is a transition matrix for an MC.

amples

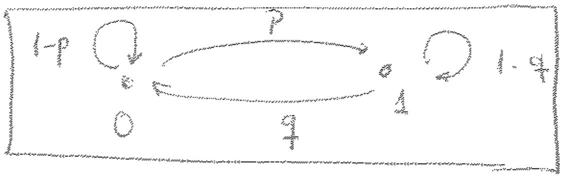
) Two-state MC - A (highly unrealistic) weather model.

et $X_n = \begin{cases} 0 & \text{rainy} \\ 1 & \text{not rainy} \end{cases}$ during the n-th day of the measurement.

uppose that at the end of each day, ∃ prob. p to switch from rainy
) not rainy, and a prob. q to switch from not rainy to rainy.

en the stoch. matrix P is $\boxed{P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}}$.

he chain can be represented by



) Finite Capacity queuing system

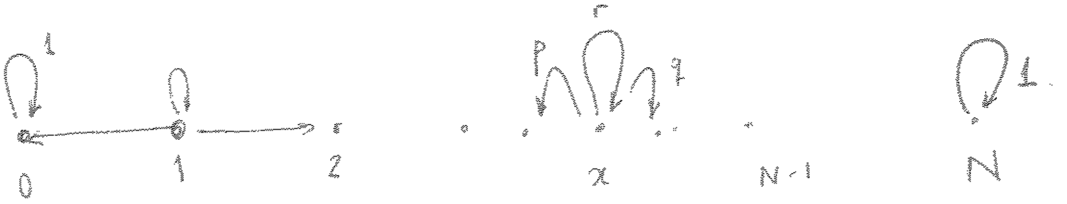
u are in a bank, where the teller serves at most one person in a given time nterval with prob. p. If the queue length in front of a teller is two, no more people may enter the queue. If there is room, at most one client attempts to join the queue with prob. q during each time interval.

- Is the length of a queue a MC?
- Find the stochastic matrix. (the transition matrix).

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1-q & q & 0 \\ 1 & (1-q)\cdot p & \underbrace{1-(1-p)q-(1-q)p} & (1-p)q \\ 2 & 0 & p & 1-p \end{array}$$

$$\longrightarrow \underbrace{pq}_{\substack{\text{served}\\+\\\text{entered}}} + \underbrace{(1-p)(1-q)}_{\substack{\text{not}\\\text{served}\\\text{not}\\\text{entered}}}$$

) Gambler's Ruin Problem (Random walk with absorbing boundaries)



$\boxed{r+p+q=1}$. For $1<x<N$, $p(x,x)=r$, $p(x,x-1)=p$, $p(x,x+1)=q$.

Why are the transition matrices useful?

Ve need to find $\boxed{\mathbb{P}\{X_n = j \mid X_0 = i\}}$ (††)(Look deep into the future!)

The only information we have are ϕ_0 (initial distribution) and P (the tr. m.).

(††) is called the <u>n- step transition</u>.

$$\boxed{p_n(i,j) = \mathbb{P}\{X_n = j \mid X_0 = i\} \doteq \mathbb{P}\{X_{n+k} = j \mid X_k = i\}}$$

ssum $\overline{\Phi}_0 = (\phi_0(0), \dots, \phi_0(N))$ i.d.

hen, by law of total probability

$$\underbrace{P_\phi\{X_n = j\}}_{\text{call this } \phi_n(j)} = \sum_{i \in S} \phi_0(i)\, \underbrace{\mathbb{P}\{X_n = j \mid X_0 = i\}}_{\text{Need to know this!}}.$$

laim : $p_n(i,j) = (P^n)_{i,j}$ = the (i,j) entry of matrix P^n.

roof : We use induction.

For $n = 1$ true!

Assume $p_n(i,j) = (P^n)_{i,j}$ $\forall (i,j)$. We'll show it for $n+1$.

Then
$$\mathbb{P}\{X_{n+1} = j \mid X_0 = i\} \underset{\text{why?}}{\doteq} \sum_{k \in S} \mathbb{P}\{X_n = k \mid X_0 = i\}\, \mathbb{P}\{X_{n+1} = j \mid X_n = k\}$$
$$= \sum_{k \in S} p_n(i,k) \cdot p(k,j)$$
$$\underset{\text{why?}}{=} (P^n \cdot P)_{i,j} \quad \square$$

on, if $\overline{\Phi}_n = (\phi_n(1), \dots, \phi_n(N)) \Rightarrow \overline{\Phi}_n = \overline{\Phi}_0 P^n$

Lec. 3

Recap.

The n-th step transition $P\{X_n = j \mid X_0 = i\} = P_n(i,j)$ is the i-jth coordinate of the matrix P^n, P is transition matrix.

$$\mathbb{P}\{X_n = j\} \overset{!}{=} \sum_{k=1}^{N} \underbrace{P\{X_n = j \mid X_0 = k\}}_{P_n(k,j)} \cdot \underbrace{\mathbb{P}\{X_0 = k\}}_{\substack{\phi_0(k) \\ \text{initial distribution}}} . \quad (\dagger)$$

Let $\overline{\phi_0}$ be the vector $[\phi_0(1), \dots, \phi_0(N)]$ that gives the initial distribution of the chain.

Then $(\dagger)$ becomes (assuming $\overline{\phi_n} = [\mathbb{P}\{X_n=1\}, \mathbb{P}\{X_n=2\}, \dots, \mathbb{P}\{X_n=N\}]$)

$$\boxed{\overline{\phi_n} = \overline{\phi_0} \cdot P^n}$$

- vector-matrix form!

xample. (Two-state MC)

Let $P = \begin{array}{cc} S & R \\ \left[\begin{array}{c} 3/4 \\ 1/6 \end{array}\right. & \left.\begin{array}{c} 1/4 \\ 5/6 \end{array}\right] \end{array}$. Let $\phi_0 = [1, 0]$ (⇒ Meaning?)

What is the probability that during the 6-th day is sunny?

Answer

$$P^6 = \begin{bmatrix} 3/4 & 1/4 \\ 1/6 & 5/6 \end{bmatrix}^6 \overset{!}{=} \begin{bmatrix} .424 & .576 \\ .384 & .616 \end{bmatrix}. \quad \text{Then } \overline{\phi_6} = \overline{\phi_0}\, P^6 \dots$$

$$= [1, 0] \begin{bmatrix} .424 & .576 \\ .384 & .616 \end{bmatrix}$$

$$= [\underline{.424}, .576].$$

Now, assume you want to settle a city. You are the first settler and you bserve this weather pattern! What is the probability that it will be sunny r rainy in two years time?

arge time behavior and invariant probability

- Two ways to do it.
  1) Using Linear algebra
  2) Using probabilistic methods + finite sums!

Method 2. (Becomes really difficult when state space becomes bigger)

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}$$

We are interested in finding $\underline{\mathbb{P}\{X_{n+1}=0\}}$

aw of total probability:

$$\begin{aligned}
\mathbb{P}\{X_{n+1}=0\} &= \mathbb{P}\{X_{n+1}=0, X_n=0\} + \mathbb{P}\{X_{n+1}=0, X_{n}=1\} \\
&= \underbrace{\mathbb{P}\{X_{n+1}=0 \mid X_n=0\}}\,\mathbb{P}\{X_n=0\} + \underbrace{\mathbb{P}\{X_{n+1}=0 \mid X_n=1\}}\,.\,\mathbb{P}\{X_n=1\} \\
&= \mathbb{P}\{X_n=0\}.(1-a) + b \,.\, \mathbb{P}\{X_n=1\} \\
&= (1-a)\,\mathbb{P}\{X_n=0\} + b\,(1-\mathbb{P}\{X_n=0\})
\end{aligned}$$

$$\boxed{\mathbb{P}\{X_{\underline{n+1}}=0\} = b + (1-a-b)\,.\,\mathbb{P}\{X_{\underline{n}}=0\}}$$

Index lowered by 1 ⟹ Re-iterate!

$$\begin{aligned}
&= b + (1-a-b)\left[b + (1-a-b)\,\mathbb{P}\{X_{n-1}=0\}\right] \\
&= b + b(1-a-b) + (1-a-b)^2\,\mathbb{P}\{X_{n-1}=0\} \\
&\;\vdots \\
&= b + b(1-a-b) + \dots + b(1-a-b)^n + (1-a-b)^{n+1}\,\mathbb{P}\{X_0=0\}. \\
&= b\left[\frac{1-(1-a-b)^{n+1}}{1-(1-a-b)}\right] + (1-a-b)^{n+1}\,\mathbb{P}\{X_0=0\}
\end{aligned}$$

$$\boxed{\mathbb{P}\{X_{n+1}=0\} = \frac{b}{a+b} + (1-a-b)^{n+1}\left[\mathbb{P}\{X_0=0\} - \frac{b}{a+b}\right].}$$

$\underset{\to\infty}{n} \; \mathbb{P}\{X_{n+1}=0\} = \frac{b}{a+b}$ (no info about ϕ_0 required!)

Repeat for $\mathbb{P}\{X_{n+1}=1\} = 1 - \mathbb{P}\{X_{n+1}=0\} \Longrightarrow \lim_{n\to\infty} \mathbb{P}\{X_{n+1}=1\} = \frac{a}{a+b}$.
if $0<a+b<2$)

ethod 1. Works in general and is computationally easy!

- Reminder. An eigenvalue of a Real matrix $\underline{A}$ is a complex number λ s.t. $\exists$ real vectors $\bar{v}, \bar{u}$ so that $\bar{v}\underline{A} = \lambda\bar{v}$ or $\underline{\underline{A}}\bar{u}^T = \lambda\bar{u}$
  ($\bar{v}$ is called left ($\bar{u}$: right) eigenvector).
- Eigenvalues are roots of the polynomial $\boxed{\det|A - xI| = 0}$ (called the characteristic polynomial).

Note: P^n is easy to calculate if P is diagonal!

ny stochastic matrix has a real eigenvalue $\lambda = 1$.
Why?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn}\end{bmatrix}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} = \begin{bmatrix}\sum_{j=1}^{n} a_{1j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj}\end{bmatrix} \overset{!}{=} \begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}.$$

- $\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$ is a right eigenvector.

et's find a Left 1-eigenvector for $\begin{bmatrix}1-a & a \\ b & 1-b\end{bmatrix}$:

ssume the eigenvector is (x,y). Then:

$$x,y)\begin{bmatrix}1-a & a \\ b & 1-b\end{bmatrix} = [x,y] \Longleftrightarrow \left\{\begin{array}{l}(1-a)x + by = x \\ ax + (1-b)y = y\end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l}-ax + by = 0 \\ ax - by = 0\end{array}\right\} \Longleftrightarrow \{ax = by\} \Longleftrightarrow \left\{y = \frac{a}{b}x.\right\}$$

$$\Longrightarrow \left\{(x,y) = x\left(1, \frac{a}{b}\right)\right\}_{,\, x\in\mathbb{R}}$$ Which eigenvector to choose?

Assume further that $x+y=1 \iff x+\frac{a}{b}x=1 \iff x=\frac{b}{a+b}$

$$y=\frac{a}{a+b}.$$

So now, $0\le x\le 1$, $0\le y\le 1$, $x+y=1$ and $(x,y)=\left[\frac{b}{a+b}, \frac{a}{a+b}\right]$

ore·over

$$\left[\frac{b}{a+b}, \frac{a}{a+b}\right]\cdot P = \left[\frac{b}{\alpha+b}, \frac{\alpha}{\alpha+b}\right]$$

$$\left[\frac{b}{\alpha+b}, \frac{\alpha}{\alpha+b}\right]\cdot P^n = \left[\frac{b}{\alpha+b}, \frac{\alpha}{\alpha+b}\right]$$

hat happens if $\Phi_0=\left[\frac{b}{\alpha+b}, \frac{\alpha}{\alpha+b}\right]$? Nothing! That's why

; called <u>invariant distribution</u>.

Lec. 4

<u>latural questions</u>

)o we always have an invariant distribution? NO, what if 1-eigenvector has negative entrie?

If we have one, is it unique? Not necessarily, what if 1 is not a simple eigenvalue?

Do we care about other eigenvalues? Yes! Here is why...

<u>arge time behavior for stochastic matrix.</u>

$P=\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$. Can we say something about $\lim_{n\to\infty} P^n$?

haracteristic polynomial $p(\lambda)=\det(P-\lambda I)=\begin{vmatrix} 1-a-\lambda & a \\ b & 1-b-\lambda \end{vmatrix}$

$$=(1-a-\lambda)(1-b-\lambda)-ab$$

$$=\lambda^2-(1-a+1-b)\lambda-ab+(1-a)(1-b)$$

$$=\lambda^2-(2-a-b)\lambda+1-a-b. = (\lambda-1)(\lambda-(1-a-b))$$

genvalues $\lambda = 1$ (yay!) $\lambda = 1 - a - b$. Note that $|1-a-b| < 1$

raise P to powers, first diagonalize:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix} = Q^{-1} P Q \quad \text{, where } Q \text{ needs to be specified}$$

he collumns of the matrix Q are <u>right</u> - eigenvectors, corresponding to ne ordering of the eigenvalues:

in eigenvector for $\lambda = 1-a-b$ is $[-a,\ b]$ (Check!)

)o $$Q = \begin{bmatrix} 1 & -a \\ 1 & b \end{bmatrix} \xrightarrow{\text{(Why?)}} Q^{-1} = \begin{bmatrix} \frac{b}{a+b} & \alpha/\alpha+b \\ -1/\alpha+b & \frac{1}{\alpha+b} \end{bmatrix}$$

Then $$P^n = (QDQ^{-1})^n = QD^nQ^{-1} = Q \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} Q^{-1}$$

$$= \begin{bmatrix} \frac{b + a(1-a-b)^n}{\alpha+b} & \frac{a - a(1-a-b)^n}{\alpha+b} \\ \frac{b - b(1-a-b)^n}{a+b} & \frac{a + b(1-a-b)^n}{a+b} \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} \overline{\pi} \\ \overline{\pi} \end{bmatrix}.$$

ere $\overline{\pi}$ is invariant distribution!

<u>uestion.</u>

)o all stochastic matrices satisfy $\lim_{n\to\infty} P^n = \begin{bmatrix} \overline{\pi} \\ \vdots \\ \overline{\pi} \end{bmatrix}$?

<u>swer</u>: <u>No</u>

What can fail? Three possible problems!

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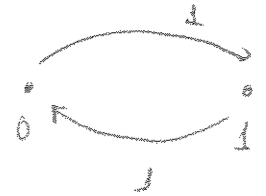


$\Longrightarrow P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I, \quad P^3 = P^2P = P, \quad P^4 = (P^2)^2 = I \quad \ldots$

$$\begin{cases} P^{2n+1} = P \\ P^{2n} = I \end{cases}$$

What makes the "theorem" fail?

We can predict the position accurately according # steps.

- Periodicity ~~(periodicity implies a lot of zeros in high powers.)~~

$P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ , $P^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P^3=\begin{pmatrix}0&1\\1&0\end{pmatrix}$: no limit!!

- $$P=\begin{bmatrix} 1/2 & 1/2 & & \\ 1/2 & 1/2 & & 0 \\ & & 1/2 & 1/2 \\ 0 & & 1/2 & 1/2 \end{bmatrix} \xrightarrow{P.P.P} P^n=\begin{bmatrix} 1/2 & 1/2 & & \\ 1/2 & 1/2 & & 0 \\ & & 1/2 & 1/2 \\ 0 & & 1/2 & 1/2 \end{bmatrix} \neq \begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

What makes the "theorem" fail?

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

st.m

st. matrix.

Chain:

no communication!

Chain can be broken down to ~~[illegible]~~ smaller chains! (⇒ Reducibility)

~~Again, there are plenty of zeroes!~~

- $$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^n \approx \begin{bmatrix} 1 & & 0 \\ 1/2 & 0 & 1/2 \\ 0 & & 1 \end{bmatrix}$$

Random walk with absorbing boundaries!

the random walk ever hits 0 or 2 it will be stuck!

there are usellss states in the limit (⇒ Transience). ~~Again, look all the zeroes!!~~

Note: if P has no zero entries, then none of those issues happens!

END OF LECTURE

Fact: If P is a stochastic matrix, such that for some n P^n has all positive entries then the left 1-eigenvector can be chosen with non-negative entries and the eigenvalue 1 is simple and all other eigenvalues are in absolute value less than 1.

Lec. 5
Math of Google
then Thws.

ection 1.3 - Classification of States:

rron - Frobenius Thm.

Suppose P is a stochastic matrix with all entries strictly positive. Then 1 is a simple eigenvalue, there exists an invariant distribution (hence unique) and all other eigenvalues has absolute value strictly less than 1.

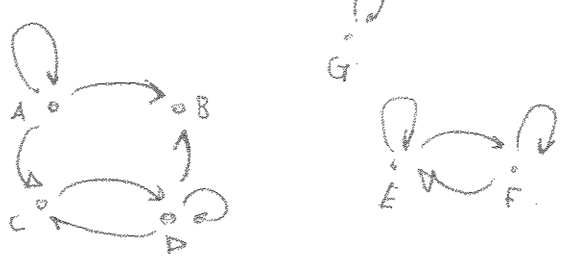
$$\Rightarrow \text{Then } \lim_{n\to\infty} \mathbb{P}^n = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}.$$

hat conditions are sufficient so that $\mathbb{P}^n$ has all positive entries or n sufficiently large?

- Anything that overcomes
  - (A) Reducibility
  - (B) Periodicity
  - (C) Transience.

) Reducibility:

Problem. The chain could split into several pieces

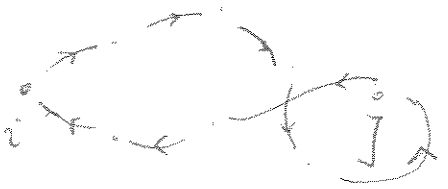


The states don't "communicate".

efinition

Two states i and j communicate with each other (written $i \longleftrightarrow j$) if there exists $m, n \geqslant 0$ such that $P_m(i,j) > 0$ and $P_n(j,i) > 0$

i.e.



$\exists$ closed circuit that contains i and j and can travel from i to j

The relation $\longleftrightarrow$ is

i) reflexive: $i \longleftrightarrow i$ $\quad (P_0(i,i) = 1 > 0)$

ii) symmetric $i \longleftrightarrow j \Longrightarrow j \longleftrightarrow i$

iii) transitive $i \longleftrightarrow k, \ k \longleftrightarrow j \Longrightarrow i \longleftrightarrow j$

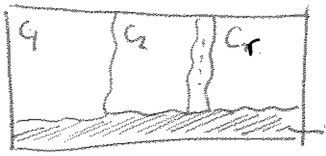
Proof of transitivity:

Assume $P_{m_1}(i,j) > 0$, $P_{m_2}(j,k) > 0$. Then

$$
\begin{aligned}
P_{m_1+m_2}(i,k) &= \mathbb{P}\{X_{m_1+m_2} = k \mid X_0 = i\} \\
&\geq \mathbb{P}\{X_{m_1+m_2} = k, X_{m_1} = j \mid X_0 = i\} \quad \text{(why?)} \\
&\overset{\substack{L.T.P \\ \mathbb{P}\{\cdot|X_0\}}}{=} \mathbb{P}\{X_{m_1+m_2} = k \mid X_{m_1} = j, X_0 = i\} \cdot \mathbb{P}\{X_{m_1} = j \mid X_0 = i\} \\
&\overset{M.P.}{=} \mathbb{P}\{X_{m_1+m_2} = k \mid X_{m_1} = j\} \cdot \mathbb{P}\{X_{m_1} = j \mid X_0 = i\} \\
&\overset{T.H}{=} \mathbb{P}\{X_{m_2} = k \mid X_0 = j\} \cdot \mathbb{P}\{X_{m_1} = j \mid X_0 = i\} \\
&= P_{m_1}(i,j)\, P_{m_2}(j,k) > 0.
\end{aligned}
$$

he relation "$\longleftrightarrow$" is an equivalence relation:

partitions the state space into communication classes : $S/_{\longleftrightarrow}$



← recurrent classes

→ The rest? Transient ~~states~~ classes

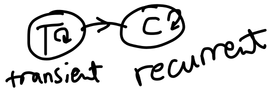
$$P = \left[\begin{array}{cc|c} P_1 \ddots & 0 & 0 \\ 0 & P_r & \\ \hline S & & Q \end{array}\right] \qquad P^n = \left[\begin{array}{cc|c} P_1^n \ddots & 0 & \\ 0 & P_r^n & \\ \hline S_n & & Q^n \end{array}\right] \leftarrow \text{Later}$$

Any matrix satisfying the Perron-Frobenius thm must have only 1 communication class! with all states in it. Such a matrix is called irreducible. (i.e. $\forall (i,j) \; \exists n,m \; P_n(i,j) > 0, P_m(j,i) > 0$.)

The communication classes are called

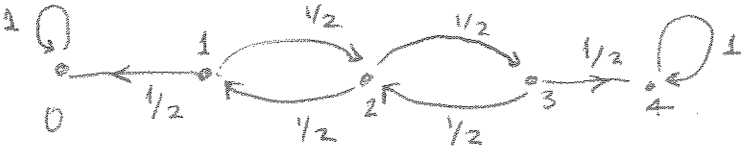
→ Transient, if the chain eventually leaves and never returns.

→ Recurrent, if the chain keeps visiting each state in the class ∞-many times.



xample.

mbler's ruin problem (Fair game)



$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Lec. 6

$\longleftrightarrow 0$

$1 \longrightarrow 0$, $1 \rightarrow 2 \longrightarrow 1 \Rightarrow 1 \leftrightarrow 2$

$2 \rightarrow 3 \longrightarrow 2 \Rightarrow 2 \leftrightarrow 3$ $\Big\}$ $1 \longleftrightarrow 3$

$\rightarrow 4$.

$\longleftrightarrow 4$

This implies transience – mass (chain) leaves $\{1,2,3\}$ comm. class.

Three communication classes!

$\{0\}$, $\{4\}$, $\{1,2,3\}$.

$\{0\} \downarrow$ Recurrent

$\{4\} \downarrow$ Recurrent

xercise 1.5.

$$= \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1/2 & 1/2 & & 0 & 0 & 0 \\ 1 & .3 & .7 & & 0 & 0 & 0 \\ 2 & 0 & 0 & .1 & 0 & .9 & 0 \\ 3 & 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 4 & 0 & 0 & .7 & 0 & .3 & 0 \\ 5 & 0 & .2 & 0 & .2 & .2 & .4 \end{array}$$

$0 \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \rightarrow 1 \Rightarrow 0 \leftrightarrow 0$

$1 \leftrightarrow 0$ $1 \leftrightarrow 1$.

$2 \longrightarrow 2 \longrightarrow 4 \longrightarrow 2 \longrightarrow 4$

$2 \leftrightarrow 4$

$2 \leftrightarrow 2$

$4 \leftrightarrow 4$

$3 \rightarrow 1, 0$

$3 \rightarrow 4$

$3 \rightarrow 5 \longrightarrow 1$

$\longrightarrow 3$

$\longrightarrow 5$

$3 \leftrightarrow 5$

Three communications classes.

$\{0,1\}$, $\{2,4\}$, $\{3,5\}$.

R R T.

chain that starts in a recurrent class always stays in it!

Periodicity:

Assume P is the matrix for an irreducible MC (otherwise work with each block)

efinition. A period of a state i, $d = d(i)$ is the largest common divisor of the integer set:

$$J_i = \{n \geq 0 ;\ p_n(i,i) > 0\}$$

Properties of J_i

f $m, n \in J_i \implies m+n \in J_i$ $\quad (P_{m+n}(i,i) \geq P_m(i,i)\, P_n(i,i) > 0)$

If d is g.c.d $\{J_i\}$, then $J_i \subset \{0, d, 2d, \dots\}$

$\exists\ M_i$ s.t. $md \in J_i \quad \forall m > M_i$ $\quad (*)$.

Exercise 1.21 ~~[illegible]~~

Define $D_i = \{x \in \mathbb{N}_0 \mid dx \in J_i\}$. Then g.c.d. $(D_i) = 1$, thus there exist $m, n \in D_i$ with g.c.d $(m,n) = 1$

Euclids algorithm implies that there exist $\alpha_0, \beta_0 \in \mathbb{Z}$ s.t. $m\alpha_0 + n\beta_0 = 1$.

Now consider the set $A = \{mx + ny \mid x, y \in \mathbb{N}_0\}$.

Because J_i (and by extension D_i) are closed under addition $A \subseteq D_i$. We'll show $(*)$ for the set A.

First note that there are two consecutive integers in A:

Let $x > |\alpha_0|$, $y > |\beta_0| \in \mathbb{N}$. Then

$N = mx + ny \in A$ and $mx + ny + 1 = m(x + \alpha_0) + n(y + \beta_0) = N+1$.

We'll show that all numbers $> N^2$ are in A.

Let $m \geq N^2$ and write $m - N^2 = kN + r$, $0 \leq r < N$

Then $m = r + N^2 + kN = \underset{\in \mathbb{N}_0}{r}\,\underline{\underline{(N+1)}} + \underbrace{(N-r+k)}_{\in \mathbb{N}_0}\,\underline{\underline{N}} \in A \subseteq D_i$

■

position: The period is a class property.

t P be ~~be~~ irreducible (so only one communication class)

et n,m such that $P_n(i,j) > 0$, $P_m(j,i) > 0$. and let

$d = g.c.d\{J_i\}$. We'll show that $d = g.c.d\ J_j$

- $P_{m+n}(i,i) \geqslant P_m(i,j)\, P_m(j,i) > 0 \Rightarrow m+n \in J_i \Rightarrow d \mid m+n$.
- $P_{m+n}(j,j) \geqslant P_m(j,i) \cdot P_n(i,j) > 0 \Rightarrow m+n \in J_j \Rightarrow$

r any other $\ell \in J_j$, $m+n+\ell \in J_j$

$P_{m+n+\ell}(i,i) \geq P_m(i,j) \cdot P_\ell(j,j) \cdot P_m(j,i) > 0 \Rightarrow m+n+\ell \in J_i \Rightarrow$

$$\left.\begin{array}{l} d \mid m+n+\ell \\ d \mid m+n \end{array}\right\} \Rightarrow d \mid \ell \quad \Rightarrow \boxed{d \leqslant g.c.d\{J_j\}}$$

Similarly, interchanging the roles of i,j we have $\boxed{g.c.d\{J_j\} \leq d}$

Later: 1. How fast do we leave a transient class?
2. what is the probability we end up in a given recurrent class?
3. What happens in a recurrent class? $\begin{cases} \text{periodic } (d \geq 2) \\ \text{aperiodic } (d=1) \end{cases}$

3.3 Irreducible, aperiodic chains

- Any irred. matrix P is aperiodic iff $d=1$. Then, $\exists$ a unique invariant probability vector $\overline{\pi}$ satisfying $\overline{\pi} P = \overline{\pi}$.
- Moreover, if $\overline{q}$ is any initial probability vector

$$\lim_{n \to \infty} \overline{q} P^n = \overline{\pi}$$

- Also $\pi(i) > 0 \quad \forall i$

Reducible or periodic chains

Theorem (No proof)

If P is irreducible with period d, P will have d eigenvalues with absolute value 1, the d complex numbers z, s.t $|z|^d = 1$

- Each is simple
- In particular, 1 is simple and there exist a unique invariant prob. $\bar{\pi}$
- Given any initial distribution $\bar{\varphi}$ for large n $\bar{\varphi} P^n$ will cycle through d different distributions but they will average to $\bar{\pi}$

$$\lim_{n\to\infty} \frac{1}{d} \left[\bar{\varphi} P^{n+1} + \dots + \bar{\varphi} P^{n+d} \right] = \bar{\pi}$$

- $\bar{\pi}$ does <u>not</u> represent the limit $\lim_{n\to\infty} P_n(j,i) \neq \pi(i)$

- It still is the average ~~amou~~ proportion of time ~~spendt~~ spend on each state!

- for each pair (i,j) there exist $0 \le r < d$ such that $P(X_{kd+r} = j \mid X_0 = i) \xrightarrow[k\to\infty]{} d\pi_j$ and $P(X_n = j \mid X_0 = i) = 0$ if $n \neq r \mod d$.

Lec. 7

Section 1.4. Return times!

$_n$ is irreducible (MC) (possibly periodic), transition matrix P

What is the ~~proportion~~ amount of time spent on state j, up to (and including) time "n"? (say we call it $Y(j,n)$)

Let $\mathbb{1}\{X_m = j\} = \begin{cases} 1 & \text{if } X_m = j \\ 0 & \text{o.w} \end{cases}$ the "indicator" function

Then $\boxed{Y(j,n) = \sum_{m=0}^{n} \mathbb{1}\{X_m = j\}}$

Is it a useful expression? Not really, since we have to know all history for it to make sense, however, we can compute expectation, or limiting behavior:

$$\lim_{n\to\infty} \frac{1}{n+1} \mathbb{E}\left(Y(j,n) \mid X_0 = i\right) = \lim_{n\to\infty} \frac{1}{n+1} \mathbb{E}\left(\sum_{m=0}^{n} 1\{X_m = j\} \mid X_0 = i\right)$$

$$\overset{!}{=} \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \mathbb{E}\left(1\{X_m = j\} \mid X_0 = i\right) \overset{!}{=} \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \mathbb{P}\{X_m = j \mid X_0 = i\}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} P_m(i,j) \overset{!}{=} \pi(j)$$

$a_n \to a \Rightarrow \frac{1}{n} \sum_{m=1}^{n} a_m \to a$ (Why?)

$\Rightarrow$ $\boxed{\text{Proportion of time spent on state } j \text{ is } \pi(j)}$!

- <u>Time of first return</u>

Fix a state i, assume that $\mathbb{P}\{X_0 = i\} = 1$.

Define the firs return time

$$\boxed{T_{ii}^{1} = \min\{n \geqslant 1 : X_n = i\}}$$

Recursively, define the time of the k-th return to state i

$$\boxed{T_{ii}^{k} = \min\{n \geqslant T_{ii}^{k-1} + 1 : X_n = i\}}, \quad k \geqslant 2$$

Observations:

- Markov property implies that $\boxed{T_{ii}^{k} - T_{ii}^{k-1} \overset{\mathcal{D}}{=} T_{ii}^{1}}$ (why?)
- S.M.P. implies $\boxed{T_{ii}^{k} - T_{ii}^{k-1} \underset{\text{independent}}{\perp\!\!\!\perp} T_{ii}^{1}}$ (why?)

↳ why strong? necessary? !!

$$\frac{1}{k}\mathbb{E}(T_{ii}^{k}) = \frac{1}{k}\mathbb{E}\Big(\underbrace{T_{ii}^{1} + (T_{ii}^{2} - T_{ii}^{1}) + \cdots + (T_{ii}^{k} - T_{ii}^{k-1})}_{\text{Sum of } k\text{-many i.i.d's}}\Big)$$

$$\xrightarrow[\text{Large \#'s}]{\text{Law of}} \mathbb{E}(T_{ii}^{1})$$

Handwaving argument:

There are about k visits to state i in $\underline{k\,\mathbb{E}(T_{ii}^{1})}$ steps

But also, we expect $\pi(i)\, k\, \mathbb{E}(T_{ii}^{1})$ proportion of time spent there!

Hence: $\pi(i)\, k\, \mathbb{E}(T_{ii}^{1}) = k \Rightarrow \boxed{\mathbb{E}(T_{ii}^{1}) = \frac{1}{\pi(i)}}$

Lec. 8

oof (Exercise 1.15)

)efine $r(j) = \mathbb{E}\left[\sum_{n=0}^{T_{ii}^1-1} 1\{X_n=j\}\right]$, the expected # of

isits to state j, before reaching state i for the first time!

-et $\bar{r} = [r(1), r(2), \dots, r(N)]$.

laim 1 $\bar{r}.P = \bar{r}$

Proof of Claim 1:

$$r(j) = \mathbb{E}\left[\sum_{n=0}^{T_{ii}^1-1} 1\{X_n=j\}\right] \overset{!}{=} \mathbb{E}\left[\sum_{n=1}^{T_{ii}^1} 1\{X_n=j\}\right]$$

$$\overset{!}{=} \mathbb{E}\left[\sum_{n=1}^{\infty} 1\{X_n=j, \ T_{ii}^1 \geqslant n\}\right]$$

$$\overset{!}{=} \sum_{n=1}^{\infty} \mathbb{P}\{X_n=j, T_{ii}^1 \geqslant n\}$$

$$\overset{!}{=} \sum_{k\in S} \sum_{n=1}^{\infty} \mathbb{P}\{X_n=j, X_{n-1}=k, T_{ii}^1 \geqslant n\}$$

$$= \sum_{\substack{k\in S\\ k\neq i}} \sum_{n=1}^{\infty} \mathbb{P}\{X_n=j, X_{n-1}=k, T_{ii}^1 \geqslant n-1\}$$

$$= \sum_{\substack{k\in S\\ k\neq i}} \sum_{n=1}^{\infty} \mathbb{P}\{\underset{\substack{\shortparallel\\ f(X_n)\\ \uparrow\\ \text{past}}}{X_n=j}, \underset{\substack{\shortparallel\\ f(X_0,X_1,\dots,X_{n-2})\\ \text{future}}}{T_{ii}^1 \geqslant n-1} \mid \underset{\substack{\shortparallel\\ n-1\\ \text{present}}}{X_{n-1}=k}\}\cdot\mathbb{P}\{X_{n-1}=k\}$$

$$\overset{!}{=} \sum_{\substack{k\in S\\ k\neq i}} \sum_{n=1}^{\infty} \mathbb{P}\{X_n=j \mid \overset{\text{indep.}}{X_{n-1}=k}\} \ \mathbb{P}\{T_{ii}^1 \geqslant n-1 \mid X_{n-1}=k\}\cdot\mathbb{P}\{X_{n-1}=k\}$$

$$= \sum_{\substack{k\in S\\ k\neq i}} \sum_{n=1}^{\infty} p(k,j)\ \mathbb{P}\{X_{n-1}=k, \ T_{ii}^1 \geqslant n-1\}$$

$$\overset{!}{=} \sum_{k\in S} \sum_{n=1}^{\infty} p(k,j)\cdot \mathbb{P}\{X_{n-1}=k, \ T_{ii}^1 \geqslant n\}$$

$$= \sum_{k\in S} \sum_{n=1}^{\infty} p(k,j)\ \mathbb{P}\{X_{n-1}=k,\ T_{ii}^{1} \geqslant n\}$$

$$= \sum_{k\in S} p(k,j)\ \mathbb{E}\left(\sum_{n=1}^{\infty} 1\{X_{n-1}=k,\ T_{ii}^{1}\geqslant n\}\right)$$

$$= \sum_{k\in S} p(k,j)\ \mathbb{E}\left(\sum_{m=0}^{\infty} 1\{X_{m}=k,\ T_{ii}^{1}-1\geqslant m\}\right)$$

$$= \sum_{k\in S} p(k,j)\ \mathbb{E}\left(\sum_{m=0}^{T_{ii}^{1}-1} 1\{X_{m}=k\}\right)$$

$$= \sum_{k\in S} p(k,j)\ r(k) = (\bar{r}P)_j$$

← easier to start here and work it out backward!

$$\Longrightarrow \boxed{\bar{r}P = \bar{r}}$$

Claim 2

$$\mathbb{E}(T_{ii}^{1}) = \sum_{j\in S} r(j)$$

Proof of claim 2

$$T_{ii}^{1} = \sum_{n=0}^{T_{ii}^{1}-1} 1\{X_n=1\} + \sum_{n=0}^{T_{ii}^{1}-1} 1\{X_n=2\} + \cdots + \sum_{n=0}^{T_{ii}^{1}-1} 1\{X_n=N\}$$

$$= \sum_{j\in S} \sum_{n=0}^{T_{ii}^{1}-1} 1\{X_n=j\}$$

$$\Rightarrow \mathbb{E}(T_{ii}^{1}) = \sum_{j\in S} \mathbb{E}\left(\sum_{n=0}^{T_{ii}^{1}-1} 1\{X_n=j\}\right) = \sum_{j\in S} r(j).$$

Claim 3

$$\mathbb{E}(T_{ii}^{1}) = (\pi(i))^{-1}.$$

Proof of Claim 3.

$$\exists\ \lambda>0 \text{ s.t. } \lambda\bar{\pi} = \bar{r} \quad \text{(why?)}$$

$$\Rightarrow \lambda\cdot\left(\sum_{j=1}^{N} \underset{1}{\cancel{\pi(j)}}\right) = \sum_{i=1}^{N} r(j) \quad \Rightarrow \lambda = \mathbb{E}(T_{ii}^{1})$$

$$\Longrightarrow \lambda\cdot\pi(i) = r(i) = 1 \text{ (Why?)} \Longrightarrow \pi(i) = \frac{1}{\mathbb{E}(T_{ii}^{1})}$$

∎

ection 1.5. Transient States!

A state is transient if the walk cannot visit infinitelly often

i.e. $\boxed{\mathbb{E}\left(\sum_{n=0}^{\infty} 1\{X_n = j\} \right) < \infty \quad \text{iff } j \text{ is transient.}}$

et P be the matrix for the chain X_n. Assume

[illegible], j transient and $X_0 = i$, i also transient

Then:

$$\mathbb{E}\Big(\underbrace{\sum_{n=0}^{\infty} 1\{X_n = j\}}_{Y_j} \Big| X_0 = i \Big) = \sum_{n=0}^{\infty} \mathbb{P}\{X_n = j \mid X_0 = i\}$$

$$= \sum_{n=0}^{\infty} P_n(i,j)$$

Thus, $\boxed{\mathbb{E}(Y_j \mid X_0 = i) = \left(I + P + P^2 + \cdots + P^n + \cdots \right)_{i,j}}$

omputationally, this is hard! P could be very, very large

$\Longrightarrow$ <u>Need for a canonical form.</u>

Canonical form of matrix P:

$$P = \begin{array}{c|cc} & \text{Recurrent} & \text{Transient} \\ \hline R & \tilde{P} & 0 \\ S & S & Q \end{array} \quad \Longrightarrow \quad P^n = \left[\begin{array}{c|c} \tilde{P}^n & 0 \\ \hline \underset{\text{junk}}{\underset{\parallel}{S_n}} & Q^n \end{array} \right]$$

Hence, (i,j) entry of $\underline{I + P + \cdots}$ is (i,j) entry of $\underline{I + Q + Q^2 + \cdots}$

Example.

$$P_1 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$R_1 = \{0\}$, $R_2 = \{4\}$, $T_1 = \{1,2,3\}$.

$$\Longrightarrow \quad P = \begin{array}{cc|cc|ccc} & & R_1 & R_2 & \multicolumn{3}{c}{\overbrace{\quad\quad}^{T_1}} \\ & & 0 & 4 & 1 & 2 & 3 \\ \hline R_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ R_2 & 4 & 0 & 1 & 0 & 0 & 0 \\ \hline T_1 & 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ & 2 & 0 & 0 & 1/2 & 0 & 1/2 \\ & 3 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}$$

$$\tilde{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\tilde{Q} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}$$

To compute $\mathbb{E}(Y_3 \mid X_0 = 1)$ we must find the

$(1 + P_1 + P_1^2 + \cdots)_{1,3}$ entry (5×5 matrix $\Longleftrightarrow$ evil) or

$(1 + Q + Q^2 + \cdots)_{1,3}$ entry.

$$q^2 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2^2} & 0 & \underline{\underline{\frac{1}{2^2}}} \\ 0 & 1/2 & 0 \\ 1/2^2 & 0 & 1/2^2 \end{bmatrix}$$

$$^3 = \begin{bmatrix} 1/2^2 & 0 & 1/2^2 \\ 0 & 1/2^2 & 0 \\ 1/2^2 & 0 & 1/2^2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2^2 & \underline{\underline{0}} \\ 1/2^3 & 0 & 1/2^3 \\ 0 & 1/2^2 & 0 \end{bmatrix}$$

$$^4 = \begin{bmatrix} 0 & 1/2^2 & 0 \\ 1/2^3 & 0 & 1/2^3 \\ 0 & 1/2^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2^3 & 0 & \underline{\underline{1/2^3}} \\ 0 & 1/2^3 & 0 \\ 1/2^3 & 0 & 1/2^3 \end{bmatrix}$$

$$(I + Q + \cdots + Q^n + \cdots)_{1,3} = \frac{1}{2^2} + 0 + \frac{1}{2^3} + 0 + \frac{1}{2^4} + \cdots = \frac{1/2^2}{1 - 1/2} = \frac{1}{2} \;//$$

= expected # visits to state 3 (starting from 1) before absorption.

omputing infinitely many powers of Q (if we cannot find a pattern) ould potentially be annoying : <u>What do we do?</u>

- Q, Q^{n} is sub-stochastic ; $\exists$ at least one row with sum of coordinates < 1.

- Since Q contains transient states $Q^{n} \xrightarrow[n \to \infty]{} 0$ (why?)

- This implies that all eigenvalues $\{|\lambda_i|\}_{1 \le i \le m} < 1$. (why?)

- In turn, this implies that $\underline{I - Q}$ is invertible !

  Proof. $\det(I - Q) = (1-\lambda_1)(1-\lambda_2) \cdot (1-\lambda_m) \neq 0$!

Now observe that:

$$(I + Q + Q^2 + \cdots)(I - Q) = I.$$

$$\Rightarrow \boxed{(I + Q + Q^2 + \cdots) = M = (I - Q)^{-1}}$$

n the previous example:

$$\boxed{M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 3/2 & 1 & 1/2 \\ 2 & 1 & 2 & 1 \\ 3 & 1/2 & 1 & 3/2 \end{array}}$$

Question. What is the expected number of steps before absorption if we start from 1? from 2?

Answer. Starting from 1 is:

# of visits ~~fro~~ to 1 + # visits to 2 + # visits to 3

$3/2 + 1 + 1/2 = 3$

Starting from 2

# of vists to 2 + # visits to 3 + # visits to 1 $= 4$.

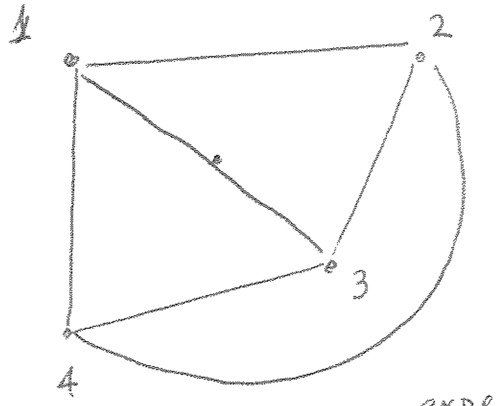
$2 + 1 + 1$

Lec. 9

This "technique" can be used to determine the expected number of steps to visit state j starting from state i even if they are recurrent states

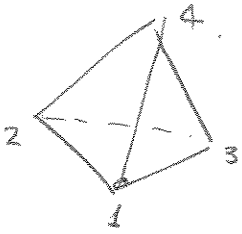
How? We make the chain transient with one absorbing class !

Example: Random walk on a tetrahedron



$p(i,j) = 1/3 \quad i \neq j$

$p(i,i) = 0.$



Q: What is the ~~probability~~ expected ~~to~~ number of steps to visit state 4 if we start from state 1 ?

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/3 & 1/3 & 1/3 \\ 2 & 1/3 & 0 & 1/3 & 1/3 \\ 3 & 1/3 & 1/3 & 0 & 1/3 \\ 4 & 1/3 & 1/3 & 1/3 & 0 \end{array} \xrightarrow[\text{first}]{\text{Rewrite } P \text{ with state 4}} P_1 = \begin{array}{c|cccc} & 4 & 1 & 2 & 3 \\ \hline 4 & 0 & 1/3 & 1/3 & 1/3 \\ 1 & 1/3 & 0 & 1/3 & 1/3 \\ 2 & 1/3 & 1/3 & 0 & 1/3 \\ 3 & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

$$\xrightarrow[\text{of interest absorbing}]{\text{Make state}} P = \begin{array}{c|c|ccc} & 4 & 1 & 2 & 3 \\ \hline 4 & 1 & 0 & 0 & 0 \\ \hline 1 & 1/3 & 0 & 1/3 & 1/3 \\ 2 & 1/3 & 1/3 & 0 & 1/3 \\ 3 & 1/3 & 1/3 & 1/3 & 0 \end{array} \implies Q = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 \end{bmatrix}$$

$\Rightarrow$ Chain is transient now

$$\underrightarrow{\text{compute matrix } M} \qquad M: (I-Q)^{-1} \qquad I-Q = \begin{bmatrix} 1 & -1/3 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & 1 \end{bmatrix} \longrightarrow \det(I-Q) = \frac{16}{27}$$

$$\Longrightarrow \quad M = (I-Q)^{-1} = \frac{27}{16} \begin{bmatrix} 8/9 & 4/9 & 4/9 \\ 4/9 & 8/9 & 4/9 \\ 4/9 & 4/9 & 8/9 \end{bmatrix}$$

$$\Longrightarrow \quad M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3/2 & 3/4 & 3/4 \\ 2 & 3/4 & 3/2 & 3/4 \\ 3 & 3/4 & 3/4 & 3/2 \end{array} \quad \xrightarrow[\text{coordinates of starting row: (here the first)}]{\text{Sum}} \quad 3/2 + 3/4 + 3/4 = \underline{3}$$

uestion 2

Lec. 10

If we have more than 1 recurrent class (say k of them) $\{R_1, \dots, R_k\}$, what is the probability that we visit (the chain enters) R_1 ? (a specific recurrent class?) if

ethod.

Assume that all recurrent classes are singletons $\{r_i\} = R_i$, $1 \le i \le k$. (This is without loss of generality !! why?) and assume we have s transient states, $\{t_1, \dots, t_s\}$.

The canonical form of the matrix
$$P = \left[\begin{array}{c|c} I_k & 0 \\ \hline \underset{s \times k}{S_{s\times k}} & \underset{\to s\times s}{Q} \end{array}\right]$$

efine $\alpha(t_i, r_j) = P\{\exists n: X_n = r_j \mid X_0 = t_i\}$

$\alpha(r_i, r_j) = 0 \qquad i \neq j$

$\alpha(r_i, r_i) = 1.$

$$\alpha(t_i, r_j) = \mathbb{P}\{X_n = r_j \text{ eventually} \mid X_0 = t_i\} \left(\overset{\text{Why?}}{\overset{\downarrow}{=}} \lim_{n\to\infty} P_n(t_i, r_j) \right)$$

$$\overset{!}{=} \sum_{x\in S} \mathbb{P}\{X_1 = x \mid X_0 = t_i\}\, \mathbb{P}\{X_n = r_j \text{ eventually} \mid X_1 = x\}$$

$$\boxed{\alpha(t_i, r_j) = \sum_{x\in S} p(t_i, x)\,\alpha(x, r_j)} \qquad (\dagger)$$

Let $A = (a_{ij})_{\substack{1\le i\le s \\ 1\le j\le k}} = (\alpha(t_i, r_j))_{i,j}$

$$\Longrightarrow \quad \boxed{A = S_{s\times k} + Q_s A}$$

<u>Mini Proof</u>

$$\alpha(t_i, r_j) = p(t_i, r_j) + \sum_{x\in T} p(t_i, x)\,\alpha(x, r_j) \overset{!}{=} \sum_{x\in S} p(t_i, x)\,\alpha(x, r_j)$$

$$I\cdot A - QA = S \iff (I-Q)A = S \iff A = (I-Q)^{-1} S$$

$$\iff \boxed{A = \underset{s\times s}{M}\ \underset{s\times k}{S} \in \mathbb{R}^{s\times k}}$$

xample 1 Random walk with absorbing boundary.

$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{cc|ccc} 0 & 4 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{array}$$

$S \qquad Q$

$$M = \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} \Rightarrow MS = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{cc} 0 & 4 \\ 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{array}$$

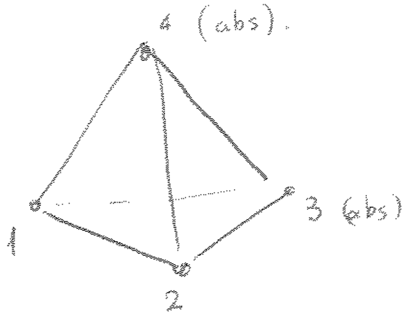
$\Longrightarrow \mathbb{P}\{\text{absorption at } 0 \text{ when we start at } 3 = \frac{1}{4}\}$ ∎

xample 2

andom walk on a tetrahedron with absorbing tip

$$S = \begin{array}{c} 4 \\ \left[\begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array}\right] \end{array}, \quad M = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3/2 & 3/4 & 3/4 \\ 3/4 & 3/2 & 3/4 \\ 3/4 & 3/4 & 3/2 \end{array}\right] \end{array} \Longrightarrow MS = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \overset{||}{\pi}$$

Random walk on a tetrahedron with <u>two</u> absorbing tips!



$$P = \begin{array}{c|cc|cc} & 3 & 4 & 1 & 2 \\ \hline 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ \hline 1 & 1/3 & 1/3 & 0 & 1/3 \\ 2 & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

$$\underbrace{\begin{matrix} 1/3 & 1/3 \\ 1/3 & 1/3 \end{matrix}}_{S} \qquad \underbrace{\begin{matrix} 0 & 1/3 \\ 1/3 & 0 \end{matrix}}_{Q}$$

$$I - Q = \begin{bmatrix} 1 & -1/3 \\ -1/3 & 1 \end{bmatrix} \Longrightarrow M = (I-Q)^{-1} = \frac{9}{8} \cdot \begin{bmatrix} 1 & 1/3 \\ 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 9/8 & 3/8 \\ 3/8 & 9/8 \end{bmatrix}$$

$$\Longrightarrow MS = \begin{bmatrix} 9/8 & 3/8 \\ 3/8 & 9/8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \left[\begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array}\right] \end{array}$$

Does it make sense?

xample 3

Covered in Lec. 9

Random walk on graphs (finite).

- Assume the graph is connected so the walk is irreducible
- Let e denote total number of edges
- $d(v)$ = degree of v = # edges that contain v.
- $v \sim w \iff \exists$ edge between v & w.
- $p(v, w) = \dfrac{\text{\# edges connecting } v \,\&\, w}{}$

For simplicity, assume that the graph has no double edges (i.e. $\exists !$ edge $v \longleftrightarrow w$).

- <u>Claim 1</u>. The invariant probability $\pi(v) = \dfrac{d(v)}{2e}$.

<u>Proof</u>: Since it is given, we need to check that

$$\sum_{i=1}^{N} \pi(v_i) \cdot p(v_i, v_j) = \pi(v_j)$$

~~$\sum_{i=1}^{N} \pi(v_i)$~~

$$\sum_{v_i \sim v_j} \pi(v_i) . p(v_i, v_j) =$$

$$= \sum_{v_i : v_i \sim v_j} \pi(v_i) \frac{1}{d(v_i)} = \sum_{v_i : v_i \sim v_j} \frac{d(v_i)}{2e} \cdot \frac{1}{d(v_i)}$$

$$\overset{!}{=} \frac{d(v_j)}{2e} = \pi(v_j) \implies \underline{\text{Yay!!}}$$

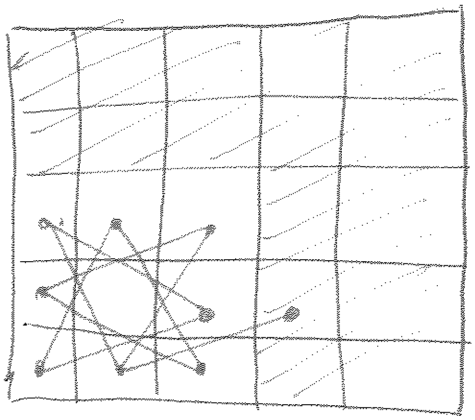
Also:

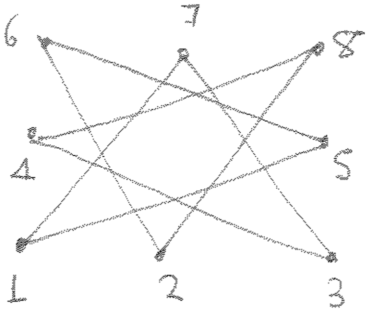
$$\sum \pi(v_i) = \frac{1}{2e} \overbrace{\sum d(v_i)}^{\text{\# of edges but each edge counted twice}} = \frac{2e}{2e} = 1$$

<u>Questions we can answer (even if not obvious)</u>

- Random chessboard knight: (3x3 otherwise is long)

to count e directly would take a while. But for a given v, counting d(v) is easy. and $e = \frac{1}{2}\sum d(v)$.





. If we start at 1, when can we ~~ever~~ expect to return?

$$\mathbb{E}(T_{11}^{1}) = \frac{1}{\pi(1)} \overset{!}{=} \left(\frac{d(v)}{2e}\right)^{-1} = \frac{2e}{d(v)} = e = 8$$

samples

Lec. 12

Section 0.3. Linear Difference Equations

$$*)\quad \begin{cases} f(n) = a f(n-1) + b f(n+1) & k<n<N \quad (N\in \mathbb{N}\cup\{+\infty\}) \\ f(k) = c_0 \\ f(k+1) = c_1 \end{cases}$$

: we know $(*)$, we can solve it recursively by the formula

$$\boxed{f(n+1) = \frac{1}{b}\left[f(n) - a f(n-1)\right]}$$

Guess solutions (as in Diff. Eq theory)

Assume $f(n) = u^n \Longrightarrow \quad u^n = a u^{n-1} + b u^{n+1} \iff$

$$u = a + bu^2 \iff \underbrace{bu^2 - u + a}_{\text{characteristic polynomial of difference equation}} = 0$$

$$\Longrightarrow \quad u_{1,2} = \frac{1 \pm \sqrt{1-4ab}}{2b}$$

Case I $\Delta = 1-4ab \neq 0 \Longrightarrow$ Two distinct roots , $u_1, u_2 \Longrightarrow$

$$\boxed{f(n) = \alpha_1 u_1^n + \alpha_2 u_2^n}$$

Silly remark.

If u_1, u_2 are complex numbers (conjugate),

ay $u_1 = \overline{u_2} = x+iy$, write them in polar form , $r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$$u_1 = re^{i\theta} \quad , u_2 = re^{-i\theta} \Longrightarrow \quad \boxed{f(n) = r^n\left[C_1\cos(n\theta) + C_2\sin(n\theta)\right]}$$

Case II $\Delta = 0 \iff u = \frac{1}{2b}$

Solution:

$$f(n) = \alpha_1 u^n + \alpha_2 u^n . n \qquad \text{(again, as in diff. eq)}$$

Example: Fib~~b~~onacci sequence

$a_0 = 1,\ a_1 = 1 \qquad a_{n+1} = a_n + a_{n-1} \implies$ Set $f(n) = a_n$,

$$f(n) = f(n+1) - f(n-1) \implies a = 1 ,\ b = +1 \implies$$

$$\xrightarrow[\text{of char. poly}]{\text{roots}} \quad u_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\implies \quad f(n) = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f(0) = 1 \iff 1 = \alpha_1 + \alpha_2$$

$$f(1) = 1 \iff 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

Solving

$$\left\{ \begin{aligned} &\alpha_1 + \alpha_2 = 1 \\ &\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned} \right\} \iff \left\{ \begin{aligned} &\alpha_1 + \alpha_2 = 1 \\ &\frac{1}{2} + (\alpha_1 - \alpha_2)\frac{\sqrt{5}}{2} = 1 \end{aligned} \right\}$$

$$\implies \left\{ \begin{aligned} &\alpha_1 + \alpha_2 = 1 \\ &\alpha_1 - \alpha_2 = \frac{1}{\sqrt{5}} \end{aligned} \right\} \iff \left\{ \begin{aligned} &2\alpha_1 = \frac{\sqrt{5}+1}{\sqrt{5}} \\ &\alpha_2 = 1 - \alpha_1 \end{aligned} \right\} \iff \left\{ \begin{aligned} &\alpha_1 = \frac{5+\sqrt{5}}{2.5} \\ &\alpha_2 = 1 - \frac{5+\sqrt{5}}{2.5} \end{aligned} \right\}$$

$$\Rightarrow \left\{ \begin{aligned} &\alpha_1 = \frac{5+\sqrt{5}}{10} \\ &\alpha_2 = \frac{5-\sqrt{5}}{10} \end{aligned} \right\} \implies f(n) = \frac{\sqrt{5}}{5}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \frac{\sqrt{5}}{5}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1}$$

$$\implies f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right]$$

Lec. 13

Linear recursions

$f(n) = \alpha + \beta \cdot f(n-1) \qquad K < n < N$

$f(K) = c_0$

ase I: $\boxed{\beta = 1} \Longrightarrow \left\{ \begin{array}{c} f(n) - f(n-1) = \alpha \\ f(k) = c_0 \end{array} \right\}$

$$\Longrightarrow \boxed{f(n) = n\alpha + c_0}$$

ase II $\boxed{\beta \neq 1}$

) Because of inhomogeneity $(\alpha \neq 0)$, first look for a constant solution

$$A_0 \Longrightarrow \qquad A_0 = \alpha + \beta A_0 \Longrightarrow \boxed{A_0 = \frac{\alpha}{1-\beta}}$$

) Then search for general solution $\boxed{f(n) = A_0 + A_1 u^n}$

This gives:

$$A_0 + A_1 u^n = \alpha + \beta \left[A_0 + A_1 u^{n-1} \right] \Longleftrightarrow$$

$$u^n = u^{n-1} \beta \Longleftrightarrow u = \beta$$

Thus, the solution is $\boxed{f(n) = \frac{\alpha}{1-\beta} + A_1 \cdot \beta^n}$

se initial conditions to specify A_1.

∎

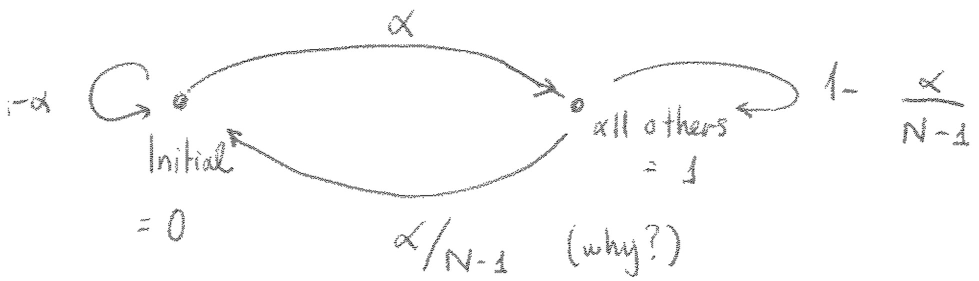
:xample 1

- Virus Mutation

Suppose that a virus can exist in N different strains and in each generation either stays the same, or mutates with probability α to a different strain, chosen uniformly at random

What is the probability that at the n-th generation, the strain is identical as the one we started.

Exploit the symmetry to reduce to a two-state MC (This almost never works)



$$\Rightarrow \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \alpha/_{N-1} & 1-\alpha/_{N-1} \end{array}$$

Ve want to compute $\underline{\underline{P_n(0,0)}}$.

$$P\{X_n=0 \mid X_0=0\} = P\{X_n=0, X_{n-1}=0 \mid X_0=0\} + \mathbb{P}\{X_n=0, X_{n-1}=1 \mid X_0=0\}$$

$$= P\{X_n=0 \mid X_{n-1}=0\}\cdot\mathbb{P}\{X_{n-1}=0 \mid X_0=0\} + P\{X_n=0 \mid X_{n-1}=1\}\,\mathbb{P}\{X_{n-1}=1 \mid X_0=0\}$$

$$\Rightarrow P_n(0,0) = p(0,0)\cdot P_{n-1}(0,0) + p(1,0)\left(1-P_{n-1}(0,0)\right)$$

$$= p(1,0) + P_{n-1}(0,0)\left[p(0,0) - p(1,0)\right]$$

$$= \frac{\alpha}{N-1} + \left(1-\alpha-\frac{\alpha}{N-1}\right)\cdot P_{n-1}(0,0)$$

$$= \frac{\alpha}{N-1} + \left(1-\frac{N\alpha}{N-1}\right)\cdot P_{n-1}(0,0)$$

se the linear inhomogeneous recursion for

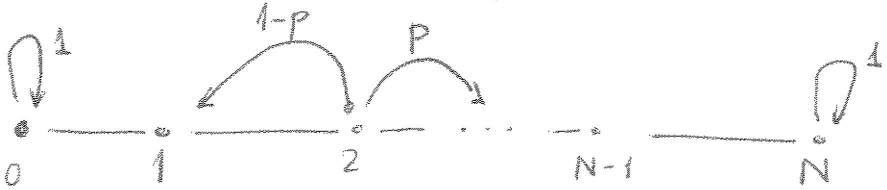
$$\alpha = \frac{\alpha}{N-1} \quad , \quad \beta = \left(1 - \frac{N\alpha}{N-1}\right) \quad , \quad P_0(0,0) = 1 \quad \text{(why?)}$$

to get

$$\boxed{P_n(0,0) = \frac{1}{N} + \left(1-\frac{1}{N}\right)\left(1-\frac{\alpha N}{N-1}\right)^n} \quad ■$$

Part of Lec. 12

xample 2 - Gambler's Ruin problem (General)



Transition probabilities:

$$P(i, i+1) = p \qquad 1 \leq i \leq N-1$$
$$P(i, i-1) = 1-p \qquad 1 \leq i \leq N-1.$$
$$P(0,0) = 1$$
$$P(N,N) = 1.$$

Define: $\boxed{\alpha(j) = \alpha(j,N) = \mathbb{P}\{\text{eventually reach } N \mid X_0 = j\}}$. Then

$\boxed{\alpha(0) = 0 \quad , \quad \alpha(N) = 1}$

Now assume $0 < j < N$. We want to compute the $\alpha(j)$'s.

One step transition method

$$\alpha(j) = \mathbb{P}\{\text{eventually reach } N \mid X_0 = j\}$$
$$= \mathbb{P}\{\text{eventually reach } N, X_1 = j-1 \mid X_0 = j\} +$$
$$\mathbb{P}\{\text{eventually reach } N, X_1 = j+1 \mid X_0 = j\}$$
$$\overset{!}{=} \mathbb{P}\{\text{eventually reach } N \mid X_1 = j-1\} \cdot \mathbb{P}\{X_1 = j-1 \mid X_0 = j\}$$
$$+ \mathbb{P}\{\text{eventually reach } N \mid X_1 = j+1\} \cdot \mathbb{P}\{X_1 = j+1 \mid X_0 = j\}$$

$$\Longrightarrow \quad \alpha(j) = (1-p)\,\alpha(j-1) + p\,\alpha(j+1)$$

Case I: $p = \frac{1}{2}$

$$\Longrightarrow \begin{cases} \alpha(j) = \frac{1}{2}\alpha(j-1) + \frac{1}{2}\alpha(j+1) , & 0<j<N \\ \alpha(0)=0 \\ \alpha(N)=1 \end{cases}$$

Solve linear difference equation $\Longrightarrow a=b=\frac{1}{2} \Longrightarrow \Delta = \sqrt{1-4ab} = 0$

$\Longrightarrow$ solution:

$$\alpha(j) = c_1 \left(\frac{1}{2b}\right)^j + c_2 \left(\frac{1}{2b}\right)^j j$$
$$= c_1 + c_2 j \quad \left(b = \frac{1}{2}\right)$$

$$\alpha(0)=0$$
$$\alpha(N)=1$$

$$\Longrightarrow \quad C_1 = 0 , \quad c_2 = \frac{1}{N} \Longrightarrow$$

$$\boxed{\alpha(j) = \frac{j}{N} \quad , p = \frac{1}{2}}$$

ase II $p \neq \frac{1}{2}$

$$\Longrightarrow \alpha(j) = \underbrace{(1-p)}_{a}\,\alpha(j-1) + \underbrace{p}_{b}\,\alpha(j+1) \Longrightarrow \Delta = \sqrt{1-4ab} = \sqrt{1-4p(1-p)}$$
$$= \sqrt{1-4p+4p^2} = \sqrt{(2p-1)^2}$$

$$\Longrightarrow \alpha(j) = C_1\left[\frac{1+(2p-1)}{2p}\right]^j + C_2\left[\frac{1-2p+1}{2p}\right]^j$$

$$\left\{\begin{array}{l} \alpha(j) = C_1 + C_2\left(\frac{1-p}{p}\right)^j \\ \alpha(0)=0 \\ \alpha(N)=1 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} C_1 + C_2 = 0 \\ C_1 + C_2\left(\frac{1-p}{p}\right)^N = 1 \end{array}\right\}$$

$$\Longrightarrow \boxed{\alpha(j) = \frac{1-\left(\frac{1-p}{p}\right)^j}{1-\left(\frac{1-p}{p}\right)^N}} \boxed{\, , \quad p \neq \frac{1}{2}}$$

Then $p \leq \frac{1}{2} \Rightarrow \frac{1-p}{p} \geqslant 1 \Rightarrow$

$$\underset{\to\infty}{\text{im}}\ \alpha(j) = \begin{cases} \lim\limits_{N\to\infty} \frac{j}{N} = 0 & p=\frac{1}{2} \\ \lim\limits_{N\to\infty} \frac{1-A^j}{1-A^N} = 0 & A>1,\ p<\frac{1}{2}. \end{cases}$$

$\Longrightarrow$ If a player has limited resources (j) and the house has a lot (unlimited), (N), then the probability of winning the house is 0 even at a fair game.

n the other hand, if $p>\frac{1}{2}$ (and so the game is in favor of the player then $\lim\limits_{N\to\infty} \alpha(j) = 1-\left(\frac{1-p}{p}\right)^j \neq 0$.

However, is not 1 either $\frac{!!}{oo}$, but the player could play forever with positive probability.

Part of Lect. 13

Question2: Suppose $p=\frac{1}{2}$. What is the estimated length of the game if we started at state j.

Answer (via one step recursion)

Define $T =$ Length of game. We want $\mathbb{E}(T \mid X_0=j)$

Let $G(j) = \mathbb{E}(T \mid X_0=j) = G(j,N)$, $G(0)=G(N)=0$.

Then:

$$G(j) = \mathbb{E}(T \mid X_0=j) ~~= \tfrac{1}{2}\mathbb{E}(T \mid X_1=j-1) + \tfrac{1}{2}\mathbb{E}(T \mid X_1=j+1)~~$$

$$= \mathbb{E}\left(T\left(1\{X_1=j-1\} + 1\{X_1=j+1\}\right) \mid X_0=j\right)$$

$$= \mathbb{E}(T\,\mathbb{1}\{X_1=j-1\} \,|\, X_0=j) + \mathbb{E}(T\,\mathbb{1}\{X_1=j+1 \,|\, X_0=j)$$

$$= \mathbb{E}(T \,|\, X_1=j-1, X_0=j)\,\mathbb{E}(\mathbb{1}\{X_1=j-1\} \,|\, X_0=j)$$

$$+ \mathbb{E}(T \,|\, X_1=j+1, X_0=j)\,\mathbb{E}(\mathbb{1}\{X_1=j+1 \,|\, X_0=j)$$

$$\overset{M.P}{=} \mathbb{E}(T \,|\, X_1=j-1)\,p(j,j-1) + \mathbb{E}(T \,|\, X_1=j+1)\,p(j,j+1)$$

$$\overset{!}{=} \mathbb{E}(T+1 \,|\, X_0=j-1)\cdot\frac{1}{2} + \mathbb{E}(T+1 \,|\, X_0=j+1)\,p(j,j+1)$$

$$= \frac{1}{2}\Big[\underbrace{\mathbb{E}(1 \,|\, X_0=j-1)}_{=1} + \underbrace{\mathbb{E}(1 \,|\, X_0=j+1)}_{=1}\Big] + \frac{1}{2}\mathbb{E}(T \,|\, X_0=j-1) + \frac{1}{2}\mathbb{E}(T \,|\, X_0=j+1)$$

$$= 1 + \frac{1}{2}G(j-1) + \frac{1}{2}G(j+1)$$ (Makes sense?)

$$\Longrightarrow \boxed{G(j) = 1 + \frac{1}{2}G(j-1) + \frac{1}{2}G(j+1)}\ (*)$$

echnically, (*) is an inhomogeneous, linear difference equation, however,

$$(*) \iff \frac{G(j)-G(j+1)}{2} = 1 + \frac{G(j-1)-G(j)}{2}$$

Let $g(j) = \dfrac{G(j)-G(j+1)}{2} \Longrightarrow \boxed{g(j) = 1 + g(j-1)}$ (1)

Solve (1) first

$$g(j) = j + C_0 \iff G(j) - G(j+1) = 2j + \tilde{C}_0$$

$$\Longrightarrow G(j+1) = G(j) - 2j + \overline{C_0}$$

$$= G(j-1) - 2(j-1) - 2j + 2\overline{C_0}$$

$$= G(j-2) - 2(j-2) - 2(j-1) - 2j + 3\overline{C_0}$$

$$\vdots$$

$$= G(0) - [2 + 2\cdot 2 + 2\cdot 3 + \dots 2j] + \overline{C_0}(j+1)$$

$$\Rightarrow \quad G(j+1) = -2\,\frac{j(j+1)}{2} + C_0\,(j+1)$$

$$\Rightarrow \quad j=N-1 \Rightarrow G(N)=0 = C_0 N - N(N-1) \Rightarrow C_0 = N-1$$

Thus: $G(j+1) = (N-1)(j+1) - j(j+1)$

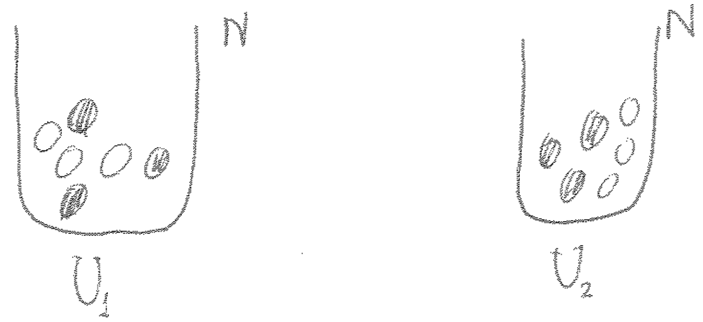
Shift the index $G(j) = (N-1)j - (j-1)j = (N-j)j$

$$\Longrightarrow \boxed{\mathbb{E}(T \mid X_0 = j) = j(N-j)}$$

Lec. 14

xample. Urn-models

1)



- Two urns, N black balls N white balls, each urn contains N balls
- At each step ~~is~~ a ball is chosen independently from each urn, uniformly at random and swap places. Let X_n = # of white balls in urn 1.

Transition probabilities:

$$\cdot\ p(\underset{\#\text{white balls}}{i}, i+1) = \mathbb{P}\left\{\begin{matrix}\text{select black}\\ \text{from } U_1\end{matrix}\right\} \cdot \mathbb{P}\left\{\begin{matrix}\text{select white}\\ \text{from } U_2\end{matrix}\right\} \quad \underline{1 \le i \le N-1}$$

$$= \frac{N-i}{N} \cdot \frac{N-i}{N} = \left(\frac{N-i}{N}\right)^2$$

$p(0,1)=1$

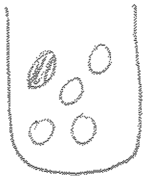
$P(N,N-1)=1$

$$\cdot\ p(i,i-1) = \frac{i}{N} \cdot \frac{i}{N} = \frac{i^2}{N^2}$$

$$\cdot\ p(i,i) = \frac{i(N-i)}{N^2} + \frac{i(N-i)}{N^2} = \frac{2i(N-i)}{N^2}$$

~~Invariant distribution:~~

Urn Model 2.



1 urn

N balls, either black or white

In each time period, a ball is chosen uniformly at random. Then with probability p, the color is ~~swapped~~ turned white and the ball returns back in the urn.

$p(0,0) = 1-p$
$p(0,1) = p$

$p(N,N) = p$
$p(N,N-1) = 1-p$

$X_n = \#$ white balls.

$1 \le j \le N-1$:

$$p(j,j+1) = \frac{N-j}{N} \cdot p \qquad , p(j,j-1) = \frac{j}{N}(1-p)$$

$$p(j,j) = \frac{N-j}{N}(1-p) + \frac{j}{N} \cdot p = \frac{N-j}{N} + p\left(\frac{j}{N} - 1 + \frac{j}{N}\right)$$

$$= \frac{N-j}{N} + p\left(\frac{2j}{N} - 1\right)$$

Claim

$$\pi(j) = \binom{N}{j} p^j (1-p)^{N-j}$$

For $j=0$:

$$\pi(0)p(0,0) + \pi(1)p(1,0) = (1-p)^N(1-p) + Np(1-p)^{N-1} \cdot \frac{1}{N}(1-p)$$
$$= (1-p)^{N+1} + p(1-p)^N = (1-p)^N = \pi(0) \quad ✓$$

Similarly for $j=N$.

Proof For $1 \le j \le N-1$:

$$(\pi P)(j) = \sum_{k=0}^{N} \pi(k) \cdot p(k,j) = \pi(j-1)p(j-1,j) + \pi(j)p(j,j) + \pi(j+1)p(j+1,j)$$

$$= \binom{N}{j-1} p^{j-1}(1-p)^{N-j+1} \frac{N-j+1}{N} p + \binom{N}{j} p^j (1-p)^{N-j} \left[\frac{N-j}{N} + p\left(\frac{2j}{N} - 1\right)\right]$$
$$+ \binom{N}{j+1} p^{j+1}(1-p)^{N-j-1} \frac{j+1}{N}(1-p)$$

$$\cancel{= \frac{(N-1)!}{(j-1)!(N-j)!} p^j (1-p)^{N-j+1} + \frac{(N-1)!}{(j-1)!(N-j)!} \frac{N}{j} p^j (1-p)^{N-j+1}}$$

$N-1$

$$= \binom{N}{j} p^j (1-p)^{N-j} \frac{j}{N} \cdot (1-p)$$

$$+ \binom{N}{j} p^j (1-p)^{N-j} \left[\frac{N-j}{N} + p \left(\frac{2j}{N} - 1 \right) \right]$$

$$+ \binom{N}{j} p^j (1-p)^{N-j} \left[\frac{N-j}{N} p \right]$$

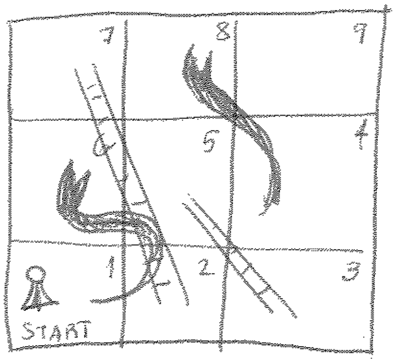
$$= \binom{N}{j} p^j (1-p)^{N-j} \left[\frac{j}{N} - \cancel{\frac{pj}{N}} + \frac{N-j}{N} + \cancel{p \frac{2j}{N}} \cancel{-p} \cancel{+p} \cancel{-\frac{jp}{N}} \right]$$

$$= \binom{N}{j} p^j (1-p)^{N-j} \qquad \text{(yay!)}$$

Does this makes sense?

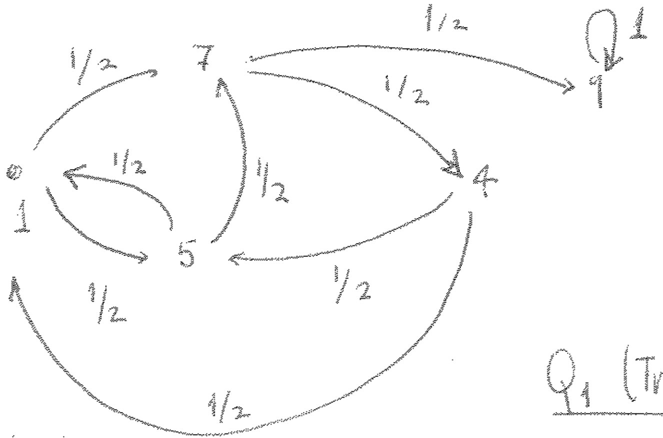
Lec. 11

xample: Snakes and ladders



- Flip a fair coin to move 1 or 2
- Ladder foot - follow to top
- Snake mouth - follow to bottom

Q_1: What is average # of turns to finish?



$$\Rightarrow \begin{array}{c|ccccc} & 1 & 4 & 5 & 7 & 9 \\ \hline 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 4 & 1/2 & 0 & 1/2 & 0 & 0 \\ 5 & 1/2 & 0 & 0 & 1/2 & 0 \\ 7 & 0 & 1/2 & 0 & 0 & 1/2 \\ 9 & 0 & 0 & 0 & 0 & 1 \end{array}$$

~~This is wrong transition matrix.~~

Why?

Q_1 (Translated) [Expected # turns until absorption

Starting from 1, is # steps in 1 + # steps in 4 # steps in 5 + # steps in 7.

$$Q = \begin{array}{cccc} 1 & 4 & 5 & 7 \\ \end{array}\begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}, \quad I - Q = \begin{bmatrix} 1 & -1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & 0 & 1 & -1/2 \end{bmatrix},$$

$$(I-Q)^{-1} = \begin{array}{c|cccc} & 1 & 4 & 5 & 7 \\ \hline 1 & 2\frac{1}{3} & 1 & 1\frac{2}{3} & 2 \\ 4 & 2 & 2 & 2 & 2 \\ 5 & 1\frac{2}{3} & 1 & 2\frac{1}{3} & 2 \\ 7 & 1 & 1 & 1 & 2 \end{array}$$

We want the sum of the first row $\Rightarrow$ $\boxed{\mathbb{E}\left[T_{finish}\right] = 7}$.

2.] What is the probability of finishing the game without being swallowed, after you reach 5?

2.] We need to go to 7 and then 9 $\Rightarrow \frac{1}{4}$.

3.] What is the probability that you play for more than 10 turns?

3.] $\mathbb{P}\{\text{play more than 10 turns} \mid X_0 = 1\}$

$$= P_{10}(1,1) + P_{10}(1,4) + P_{10}(1,5) + P_{10}(1,7)$$

$$= 1 - P_{10}(1,9) = 0\cdot 2041 \text{ (why?)}.$$

4.] What are the communication classes and their periods?

$R_1 = \{9\}$.

$T_1 = \{1, 4, 5, 7\}$

For the period: $P_2(1,1) \geqslant \frac{1}{4} > 0 \Rightarrow d \mid 2$.

$$P_3(1,1) > 0 \quad (1 \to 7 \to 4 \to 1) \Rightarrow d \mid 3$$

$$\Rightarrow d \mid \gcd(2,3) = 1 \Rightarrow d = 1.$$

■

Simulation Lecture !

- Generate uniform (0,1)

$X \sim \text{Unif}(0,1) \iff \mathbb{P}\{X \leq x\} = x \quad \forall x \in 0,1$

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

Generate Bernoulli r.v.

Bernoulli: $X = \begin{cases} 1 & \text{prob. } p \\ 0 & \text{prob } 1-p \end{cases}$

If $U \sim \text{Unif}(0,1) \Rightarrow$ To generate X, $\mathbb{P}\{X=1\} = p = \int_0^p 1\, dx$

$= \mathbb{P}\{U \leq p\}$

Steps

Call a Uniform(0,1)

- Check if it is $\leq p$ or not.
- If yes $\Rightarrow X = 1$
  no $\Rightarrow X = 0$

$\}$ → generates Bernoulli r.v.

) Discrete distributions with finitely many values

Generate X, with mass function $\mathbb{P}\{X=i\} = p_i \qquad 1 \leq i \leq N.$

$\sum_{i=1}^{N} p_i = N.$

Show that $\mathbb{P}\{X=i\} = \mathbb{P}\left\{\sum_{k=1}^{i-1} p_k < U \leq \sum_{k=1}^{i} p_k\right\}$

Note: $\sum_{i=1}^{0} f(k) = 0$!

Here is a <u>pseudocode</u>:

```
1). Generate Unif$(0,1)$ r.v. $U$
2). If $U <= p_1$ set $X = x_1$ & stop.
3). Else, set $k = 2$, ~~Prev~~ $S_{prev} = p_0$, $S = p_0 + p_1$.
4). If $S_{prev} < U \leq S$, set $X = x_k$ & stop
5). Else, set $K = k+1$, $S_{prev} = S$, $S = S + p_K$; go to (4).
```

uestion : What if X has countable (but infinitely many) states? Is there any problem?

nswer : Potentially we have an infinite loop! (4-5) repeating?

2: : Well do we?

2: No!.

Proof

Let T be the time algorithm stops! Then

$$\mathbb{P}\{T > k\} = \mathbb{P}\{\text{the algorithm does not stop after } k \text{ steps}\}$$

$$= \mathbb{P}\{U > p_1 + p_2 + \cdots + p_k\} = 1 - \mathbb{P}\{U \leq p_1 + \cdots + p_k\}$$

$$= 1 - \sum_{j=1}^{k} p_j = \sum_{j=k+1}^{\infty} p_j$$

Then $\mathbb{P}\{T = \infty\} = \lim_{k \to \infty} \mathbb{P}\{T > k\} \longrightarrow 0$ (why?).

<u>ore distributions</u>

<u>lass problem</u> 1) How to simulate a Uniform r.v. on $\{1, \ldots, n\}$?

2) Show that $X = 1 + \lfloor nU \rfloor$ is unif. distributed on $\{1, \ldots, n\}$.

Answer 1) $X=j \Longleftrightarrow \boxed{\frac{j-1}{n} < U \le \frac{j}{n}}$ $(\dagger)$

2) Multiply $(\dagger)$ by $n \Longrightarrow j-1 < nU \le j \Longrightarrow j-1 = \lfloor nU \rfloor$

$\Longrightarrow j = 1 + \lfloor nU \rfloor \Longleftrightarrow \boxed{X = 1 + \lfloor nU \rfloor}$ ∎

Geometric r.v.

$\mathbb{P}\{X=j\} = p(1-p)^j$ $\quad j=0,1,\dots$, p prob. of success

$$\Rightarrow X=j \Longleftrightarrow p\sum_{k=0}^{j-1}(1-p)^k < U \le p\sum_{k=0}^{j}(1-p)^k \qquad \text{(again, empty sum}=0)$$

$$\Rightarrow \cancel{p}\,\frac{1-(1-p)^j}{1-(1-p)} < U \le \cancel{p}\,\frac{1-(1-p)^{j+1}}{\cancel{1-(1-p)}}$$

$$\Rightarrow \begin{cases} 1-U < (1-p)^j & \Longrightarrow \dfrac{\ln(1-U)}{\ln(1-p)} \overset{!}{>} j \\[2ex] (1-p)^{j+1} \le 1-U & \Longrightarrow j+1 \ge \dfrac{\ln(1-U)}{\ln(1-p)} \end{cases}$$

$$\Rightarrow X = \left\lfloor \frac{\ln(1-U)}{\ln(1-p)} \right\rfloor . \quad \text{But } 1-U \underset{\uparrow \text{ why?}}{\overset{\mathcal{D}}{=}} \tilde{U} \Longrightarrow$$

$$\boxed{X = \left\lfloor \frac{\ln \tilde{U}}{\ln(1-p)} \right\rfloor}.$$

How to simulate a M.C. with finitely many states?

$\{X_n\}_{n\ge 0}$ M.C., ϕ_0 initial distribution, P transition matrix.

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$$P = \begin{array}{c|ccc} & x_1 & \cdots & x_N \\ \hline x_1 & & & \\ \vdots & & p(x_i, x_j) & \\ x_N & & & \end{array} \qquad \begin{array}{l} 1 \leq i \leq N \\ 1 \leq j \leq N. \end{array}$$

- Assume $X_0 = x_1$ (any fixed state!). Then,

$$\sum_{j=1}^{N} p(x_j, x_1) = 1 \Rightarrow \qquad X_2 = x_j \Longleftrightarrow \sum_{k=1}^{j-1} p(x_k, x_1) < U_1 \leq \sum_{k=1}^{j} p(x_k, x_1)$$

$$X_3 = x_m \Longleftrightarrow \sum_{k=1}^{m-1} p(x_k, y_2) < U_2 \leq \sum_{k=1}^{m} p(x_k, y_2)$$

Pseudocode.

```
1) • $X_0 = x_1$ , $P$ is given, $1 \leq i \leq N$ , $1 \leq j \leq N$.
2) • Generate a (very) large family of Unif $(0,1)$. (need to be independent)
3) {$U_1, U_2, \dots, U_M, \dots U_L$} , $1 \leq k \leq L$.
   Let $k = 1$
4) • $X_1 = x_j \Longleftrightarrow \underbrace{\sum_{k=1}^{j-1} p(x_1, x_k)}_{\text{Sprev}} < U_1 \leq \underbrace{\sum_{k=1}^{j} p(x_1, x_k)}_{S.}$
   Let $k = k+1$
5) $X_k = x_j \Longleftrightarrow \sum_{k=1}^{j-1} \underbrace{p(X_{k-1}, x_\ell)} < U_2 \leq \sum_{k=1}^{j} p(\underbrace{X_{k-1}, x_\ell});$
   Repeat until $k = L$. ▨
```

nvariant distributions Numerically. (For aperiodic, irreducible matrices)

- $\pi(j) = \lim_{n \to \infty} \dfrac{\#\text{ visits to state } j \text{ in } n \text{ steps.}}{n}.$

- When simulating a MC, set counters

$$N_0^i = 0, \quad N_k^i = N_k^i + 1 \text{ iff } X_k = i$$ ▨