MATH 5040/6810: Homework 5 (Due Monday, December 2)

Problem 1. Assume that Archie, Betty and Veronica are shopping at Smith's for the Thanksgiving dinner. It is so late in the day that they are the only customers in the store. There are two cashiers at the store (numbered 1 and 2) and they serve customers at rates β and γ per minute, respectively. Assume the trio approaches the registers in alphabetical order. So Archie and Betty go to registers 1 and 2, respectively, and Veronica waits for a register to free up and goes there.

(a) What is the expected time in minutes for Veronica to go to a register?
(b) What is the probability Veronica will go to Archie's register? Why?
(c) What is the probability Veronica waits more than 10 minutes until a register is free?
(d) What is the expected total amount of time for Veronica to exit the store?
(e) What is the probability Veronica is the last to leave?

(a) The time Veronica goes to a register is the smallest of two exponential random variables with rates β and γ. It is hence another exponential random variable with rate $\beta+\gamma$. As such, it has an average of $1/(\beta+\gamma)$.

(b) This is the probability Arhcie's time is smaller than Betty's. It equals $\beta/(\beta+\gamma)$.

(c) This is the probability the exponential variable in (a) is larger than 10. It equals $e^{-10(\beta+\gamma)}$.

(d) The time to get to some register has an average of $1/(\beta+\gamma)$. Then, if Veronica goes to Archie's register (probability of $\beta/(\beta+\gamma)$) she will spend an additional time of $1/\beta$ (on average) and if she goes to Betty's register (probability of $\gamma/(\beta+\gamma)$) she will spend an additional time of $1/\gamma$ (on average). The total time is then

$$\frac{1}{\beta+\gamma}+\frac{\beta}{\beta+\gamma}\times\frac{1}{\beta}+\frac{\gamma}{\beta+\gamma}\times\frac{1}{\gamma}=\frac{3}{\beta+\gamma}\,.$$

(e) If Veronica goes to Archie's register (probability of $\beta/(\beta+\gamma)$) then this means Betty has not finished being served yet. The memoriless property of exponential random variables says then that we can consider that Betty and Veronica both started being served when Veronica arrived at Archie's register. Given that, the probability Betty finishes first is $\gamma/(\beta+\gamma)$. A similar reasoning applies for the case when Veronica goes to Betty's register. In total, the probability Veronica is the last to leave equals

$$\frac{\beta}{\beta+\gamma}\times\frac{\gamma}{\beta+\gamma}+\frac{\gamma}{\beta+\gamma}\times\frac{\beta}{\beta+\gamma}=\frac{2\beta\gamma}{(\beta+\gamma)^2}\,.$$

Problem 2. Consider the following model for borrowing and returning books to the library. The library has c copies of a given book. Customers line up to get the book. When a copy becomes available, the next costumer in line to get the book immediately gets that copy. When a customer returns a copy, it becomes immediately available for the next customer in line. The inter-arrival times between customers who want the book are independent exponential random variables with rate $b>0$. The times customers spend between borrowing the book and returning it are independent exponential random variables with rate $r>0$.

(a) This can be viewed as a birth-and-death process. In fact, it is a queuing model. Which one?

(b) Suppose we know from past borrowing and returning records that customers join the line at the rate of 2 people per week and they return the book, on average, after 6 weeks. How many copies of the book should the library get?

(a) This is an $M/M/k$ queue with $k = c$. The books play the role of service stations. Arrivals are customers queuing up to borrow the book and departures are customers returning the book. The service time is the time between the customer getting a copy of the book and the time they return the book.

(b) The rate of arrival is given to be $\lambda = 2$ customers per week and since the average service time is 6 weeks, the rate of service is given by $\mu = 1/6$ customers per week. A stable queue is one that is positive recurrent and for that we need $k\mu > \lambda$. Plugging in we get $c/6 > 2$ and so $c > 12$. The library should thus get at least 13 copies.

Problem 3. Consider an $M/M/\infty$ queue where customers arrive at rate 4 and, without standing in line, each customer starts getting serviced immediately with the service time having rate 2.

(a) This can be viewed as a birth-and-death process. Write down the rates μ_n and λ_n for each integer $n \geq 0$.

(b) Is the chain transient, null recurrent, or positive recurrent?

(c) Find the invariant probability for the queue length. (Hint: use the formula we derived in class and try to recognize the mass function you find.)

(d) Compute the expected queue length when the chain has been going on for a long time.

(a) $\lambda_n = 4$ and $\mu_n = 2n$ for all integers $n \geq 0$.

(b) We have

$$\sum_{n\geq 0} \frac{\lambda_0 \ldots \lambda_{n-1}}{\mu_1 \ldots \mu_n} = \sum_{n\geq 0} \frac{4^n}{n!2^n} = \sum_{n\geq 0} \frac{2^n}{n!} = e^2 < \infty.$$

Therefore, the chain is positive recurrent.

(c) We have $\pi(n) = \frac{\lambda_0 \ldots \lambda_{n-1}}{\mu_1 \ldots \mu_n}\pi(0) = \frac{2^n}{n!}\pi(0)$. To get $\pi(0)$ we sum over all integers $n \geq 0$ to get $1 = \pi(0)\sum_{n=0}^{\infty} \frac{2^n}{n!} = \pi(0)e^2$. Therefore, $\pi(0) = e^{-2}$ and $\pi(n) = \frac{2^n e^{-2}}{n!}$. This is a Poisson distribution with parameter 2.

(d) After a long time the queue length will follow the invariant measure. The expected value of a Poisson random variable is equal to the parameter. So the expected length of the queue, after a long time, is 2.

Problem 4. Consider an $M/M/k$ queue where new customers arrive at rate λ and stand in line, waiting to be served. There are k servers available, each of which can serve one customer at a time. Once a server frees up, the next customer in line starts getting served immediately. While all k servers are busy, customers wait for the next free server. Each server serves at the same rate of μ.

(a) Write down the rates μ_n and λ_n for each integer $n \geq 0$.

(b) Find a condition on λ, μ, and k that guarantees the system is working properly, i.e. the process is positive recurrent.

(c) Say you join the queue and notice that all k servers are busy with customers and that there are $m-1$ customers ahead of you that are not being served yet (so a total of $k+m$ customers in the queue, including you). What is the average amount of time it will take one of the servers to free up? (Hint: the time it takes a given server to free up is an exponential random variable with rate μ. The time it takes one of the servers to free up is the minimum of all the times for each of the k servers. What is the random variable that is the minimum of k exponential random variables with rate μ each? What is its mean? Alternatively, you can think of this as the time for the size of the queue to go from $k+m$ to $k+m-1$.)

(d) Once a server frees up the next customer who was waiting starts being served immediately. Consequently, at that moment in time there will again be k servers busy, but $m-2$ customers ahead of you. So what you really calculated in Part (c) is the time for you to go from being m-th in line to being number $m-1$, waiting to be served. Compute now the average amount of time it will take you from the time you joined the queue to the time you start being served (i.e. the time when you are number 1 in line AND a server opens up).

Note: Combining Problem 2 and part (d) in Problem 4, you now have a formula for the average expected waiting time for a book, if you know the total number of copies the library has, the rate at which new customers queue up, the rate at which customers return the book, and how many customers you have ahead of you. Libraries can use this formula to give estimated waiting times. Similar computations can be done to give estimated waiting times to reach a customer support agent while you wait over the phone.

(a) $\lambda_n = \lambda$ for all integers $n \geq 0$. $\mu_n = n\mu$ for all integers n between 0 and k and $\mu_n = k\mu$ for all integers $n \geq k$.

(b) We worked out in class that for an $M/M/k$ queue, the condition for positive recurrence is that $k\mu > \lambda$.

(c) This is the average time it takes to move from position $k+m$ to $k+m-1$. The rate for this is $k\mu$ and thus the average time it takes equals $1/(k\mu)$. (Alternatively, you can think of this as the minimum of k exponential random variables (one for each busy server) and this is another exponential random variable with rate $k\mu$.)

(d) The calculation in (c) can be repeated m times (and for each time, we are moving from position x to $x-1$ and $x \geq k$, so the rate is $k\mu$ every time). Thus, the total time is on average equal to $m/(k\mu)$. This is the formula one can use to estimate the amount of time it will take to receive a book from the library. Here is how to see this intuitively: m is your position in the queue. Since there are k books, your position in the queue is really m/k. (Think of you being m-th in a single line with k cashiers. If customers are asked to form k lines instead, then there will be about m/k customers head of you.) Now, each customer takes on average $1/\mu$ time units to be served. So you will get the book after about $(m/k) \cdot (1/\mu)$ time units, like our formula tells us.