MATH 5040/6810: Homework 3 (Due Monday, October 21)

Solutions are in blue ink. Red ink is for alternate solutions.

Problem 1

Prove that the simple symmetric random walk is recurrent in two dimensions. (Hints: (1) To use Pólya's criterion, you need to show that a certain series has an infinite sum. It is enough for that to show that the sum along multiples of four is already infinite. (2) Starting at 0, the walk can return to 0 in $4m$ steps by moving first along one dimension, returning to 0 in $2m$ steps, and then doing the same in the other dimension.)

Solution: Following the computation in one and three dimensions, we will show that the average number of returns to 0 is infinite. That is,

$$\sum_{n=0}^{\infty} p_{2n}(0,0) = \infty.$$

(The sum is only over even number of steps because we know that the probability to return to 0 in an odd number of steps is 0.)

Note that it is enough to prove that summing the above along even n is enough. If that sum is infinite, then so will be the sum along all n. In other words, it is enough to show that

$$\sum_{m=0}^{\infty} p_{4m}(0,0) = \infty.$$

One way for the chain to go from 0 back to 0 in $4m$ steps is to go from 0 to 0 in $2m$ steps using only right and left moves, and then to go from 0 back to 0 in $2m$ steps using only up and down moves. Since this is one possible scenario (and there are many other possibilities), the probability $p_{4m}(0,0)$ is at least as big as the probability $p_{2m}^1(0,0) \times p_{2m}^1(0,0)$, where $p_k^1(i,j)$ is the probability a one-dimensional simple symmetric walk goes from i to j in k steps. But we found already that $p_{2m}^1(0,0) \sim 1/\sqrt{\pi m}$. Since $\sum_{m \geq 1} 1/m = \infty$ (to see this, compare with integral $\int_1^{\infty} 1/x \, dx$), we get that $\sum_{m \geq 1} (p_{2m}^1(0,0))^2 = \infty$. Because $\sum_{m \geq 1} p_{4m}(0,0)$ is larger, it must be infinite as well.

Here's another (longer) way to get the same result.

To return to 0 the walk needs to take the same number of steps, say i, to the left as to the right, and the same number of steps, say j, up as down. If this happens in $2n$ steps, then $2i + 2j = 2n$. In other words, $j = n - i$. And i can be any integer between 0 and n. Once we know we want to take i steps to the right and i steps to the left, we know we want to take $n - i$ steps up and $n - i$ steps down. There are $\binom{2n}{i}$ ways to choose the steps to the right, and then $\binom{2n-i}{i}$ steps to choose the steps to the left. After that, there are $\binom{2n-2i}{j}$ ways to choose the up-steps, and the remaining steps are down-steps. If we don't have a restriction on the number of steps in each direction, then we have 4 choices for each step

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and in total 4^{2n} ways to make $2n$ steps. Therefore, recalling that $j = n - i$ we have

$$p_{2n}(0,0) = \sum_{i=0}^{n} \frac{\binom{2n}{i}\binom{2n-i}{i}\binom{2n-2i}{n-i}}{4^{2n}} = \sum_{i=0}^{n} \frac{(2n)!(2n-i)!(2n-2i)!}{(2n-i)!i!(2n-2i)!i!(2n-2i-(n-i))!(n-i)!} \times \frac{1}{4^{2n}}$$

$$= \sum_{i=0}^{n} \frac{(2n)!}{(i!(n-i)!)^2} \times \frac{1}{4^{2n}} = \frac{(2n)!}{(n!)^2} \times \frac{1}{2^{2n}} \times \sum_{i=0}^{n} \left(\frac{n!}{i!(n-i)!}\right)^2 \times \frac{1}{2^{2n}}.$$

Next, observe that

$$\sum_{i=0}^{n} \left(\frac{n!}{i!(n-i)!}\right)^2 = \binom{2n}{n}.$$

This is because to choose n objects out of $2n$ total objects you could first color any n objects with say red and the rest with blue, and then pick i objects from the red and $n - i$ objects from the blue. There are $\binom{n}{i} \times \binom{n}{n-i} = \binom{n}{i}^2$ ways to do this, and i is anything between 0 and n.

Now plug this identity back up to get

$$p_{2n}(0,0) = \left(\frac{(2n)!}{(n!)^2} \times \frac{1}{2^{2n}}\right)^2.$$

In the one dimensional case we have shown that $\frac{(2n)!}{(n!)^2} \times \frac{1}{2^{2n}} \sim 1/\sqrt{\pi n}$. Hence, $p_{2n}(0,0) \sim 1/(\pi n)$, the sum of which is infinite.

Problem 2

We know things get worse as dimension increases. So since random walk in one dimension is null recurrent, we expect the two-dimensional case to be no better. Now that you proved the walk in two dimensions is recurrent, prove that it is null recurrent. (Hint: you need to rule out positive recurrence, which means to rule out existence of invariant probability measures.)

Solution: The chain is irreducible. Thus, if it were positive recurrent, there would exist an invariant measure that gives a positive weight to each point $(i,j) \in \mathbb{Z}^2$. (The value of the invariant measure at (i,j) would be the reciprocal of the average return time to (i,j), starting from (i,j).) The chain has period 2. Thus, if there were an invariant measure, we would have $p_{2n}(0,0)$ converge to the value of the invariant measure at 0. By the computation in Problem 1 we see that $p_{2n}(0,0) \to 0$ as $n \to \infty$, which rules out the existence of an invariant measure and proves null recurrence. (Note that if you choose the shorter solution to Problem 1, then you won't have that $p_{2n}(0,0) \to 0$ and you will have to do an upper bound on $p_{2n}(0,0)$ similar to the one we did for the three-dimensional case, to deduce that it goes to 0 as $n \to \infty$.)

Another way to see that there cannot be an invariant measure is to appeal to symmetry. Write the equation that the invariant measure would satisfy:

$$\pi(i,j) = \tfrac{1}{4}\pi(i-1,j) + \tfrac{1}{4}\pi(i+1,j) + \tfrac{1}{4}\pi(i,j-1) + \tfrac{1}{4}\pi(i,j+1) \quad \text{for all } (i,j) \in \mathbb{Z}^2.$$

This equation looks the same for all points (i,j) (that is, the coefficients do not depend on i and j). Symmetry then says that the π that solves it must take the same value at all

(i, j). But then it cannot add up to 1. This rules out existence of an invariant probability measure and proves null recurrence.

If you want to do this rigorously, add over all j in the above equation and call $a(i) = \sum_j \pi(i, j)$. Then we have

$$a(i) = \tfrac{1}{4}a(i-1) + \tfrac{1}{4}a(i+1) + \tfrac{1}{4}a(i) + \tfrac{1}{4}a(i).$$

Rearranging leads to $a(i+1) - a(i) = a(i) - a(i-1) = \cdots = a(1) - a(0)$. So

$$a(i) = a(i) - a(i-1) + \cdots + a(1) - a(0) + a(0) = (a(1) - a(0))i + a(0).$$

But we also have that $\sum_i a(i) = \sum_{i,j} \pi(i, j) = 1$. Therefore, it must be the case that $a(1) = a(0)$. But then this means that $a(i) = a(0)$ for all i and then it cannot add up to 1.

Problem 3

Let p and q be two numbers in $[0, 1]$ such that $p + q \leq 1$. Consider the Markov chain on nonnegative integers that does the following: from any site $x \geq 0$ it jumps to $x + 1$ with probability p, to $x + 2$ with probability q, and to 0 with probability $1 - p - q$. Is this chain transient? Null recurrent? Positive recurrent? You need to prove whatever you claim.

Solution: If $p + q = 1$, then the chain will move to the right indefinitely. This is transient, but not the interesting case. If $p + q = 0$, then both p and q are 0 and the chain ends up absorbed at 0. This is positive recurrent, but is also not an interesting case. So let us now assume $0 < p + q < 1$.

With the above assumption, the chain is irreducible since to go from site x to site y the chain can jump from x straight to 0 and then march to the right until it reaches y. Therefore, we can focus on 0 to determine the type of chain we have.

To never come back to 0 the chain has to keep moving right. To do this for n steps has probability $(p + q)^n$. To do this forever has probability $\lim_{n \to \infty} (p + q)^n = 0$. So the chain is not transient. To decide between positive and null recurrence we search for an invariant measure or compute the average time of return to 0. Both are not hard in this example, but working with the return time is easier.

To return in n steps the walk needs to go to the right $n - 1$ times and then jump to 0 at the n-th step. This has a probability of $(p + q)^{n-1}(1 - p - q)$. Therefore, the return time is a geometric random variable with success probability $1 - p - q > 0$. Its average is then $1/(1 - p - q)$, a finite number. So the chain is positive recurrent.

Problem 4

Consider the Markov chain on $\{0, 1, 2, \dots\}$ in which from each $x \geq 0$ the chain moves from x to $x + 1$ with probability $\frac{1}{x+1}$ and moves back to 0 with probability $\frac{x}{x+1}$.
(a) Is this chain irreducible or not?
(b) Is this chain transient, positive recurrent, null recurrent?

Solution: (a) The chain is irreducible because to go from x to y the chain can jump to 0 and then march to y.

(b) The probability the chain, starting at 0, does not return to 0 after n steps is given by $1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n} = \frac{1}{n!}$. Thus, the probability the chain never returns to 0 is $\lim_{n \to \infty} \frac{1}{n!} = 0$

and the chain is recurrent. To decide on whether it is null or positive recurrent, we look at the return time to 0.

First, we note that the chain does not loop back to 0. So it takes the Markov chain at least two steps to get back to 0. Now consider $n \geq 2$. For the chain to return to 0 after exactly n steps, it needs to move forward $n-1$ times and then jump to 0. This has a probability of

$$1 \times \frac{1}{2} \times \cdots \times \frac{1}{n-2} \times \frac{1}{n-1} \times (1 - \frac{1}{n}) = 1 \times \frac{1}{2} \times \cdots \times \frac{1}{n-2} \times \frac{1}{n-1} \times \frac{n-1}{n} = \frac{1}{(n-2)!} \times \frac{1}{n}.$$

The above is $P(T_0 = n)$. So

$$E[T_0] = \sum_{n=2}^{\infty} nP(T_0 = n) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!}.$$

This sum is finite. You can see this either with the ration test $((1/(n-1)!)/(1/(n-2)!)) = 1/(n-1) \to 0 < 1)$ or because the sum is equal to $\sum_{m=0}^{\infty} \frac{1}{m!} = e^1$. Since $E[T_0] < \infty$, the Markov chain is positive recurrent.

Problem 5

Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probability $p(x, x+1) = \frac{1}{2}(1 + \frac{1}{x})$ and $p(x, x-1) = \frac{1}{2}(1 - \frac{1}{x})$ for $x \geq 2$, $p(0, 1) = p(1, 2) = \frac{3}{4}$, and $p(0, 0) = p(1, 0) = \frac{1}{4}$. Show that this chain is transient. (Hint: this is a birth-and-death process.)

Solution: This is a birth-and-death process. The transition probabilities are given by $b_k = p(k, k-1) = \frac{1}{2}(1 - \frac{1}{k}) = \frac{k-1}{2k}$ and $a_k = p(k, k+1) = \frac{1}{2}(1 + \frac{1}{k}) = \frac{k+1}{2k}$, for $k \geq 2$. To show transience, we need to check that

$$(1) \qquad \sum_{k=2}^{\infty} \frac{b_1 \dots b_{k-1}}{a_1 \dots a_{k-1}} < \infty.$$

Notice that the factor of 2 in the denominators of b_k and a_k cancels out. So we have

$$\sum_{k=2}^{\infty} \frac{b_1 \dots b_{k-1}}{a_1 \dots a_{k-1}} = \sum_{k=2}^{\infty} \frac{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{k-3}{k-2} \cdot \frac{k-2}{k-1}}{\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{k-1}{k-2} \cdot \frac{k}{k-1}} = \sum_{k=2}^{\infty} \frac{\frac{1}{k-1}}{\frac{k}{2}} = \sum_{k=2}^{\infty} \frac{2}{k(k-1)}.$$

This sum is finite. To see this, either note that for k large, the main term is equivalent to $\frac{2}{k^2}$ and the sum of these terms converges because this is like $\sum_{k=1}^{\infty} \frac{1}{k^p}$ and the power of k is $p = 2 > 1$. Or, write

$$\sum_{k=2}^{\infty} \frac{1}{2k(k-1)} = \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \frac{1}{2}.$$

Since we showed (1), the Markov chain is transient.

Problem 6 Consider the 3×3 board of Snakes and Ladders we studied in class. (See Figure) The rules are: every turn you toss a fair coin. Heads move you one step forward while tails move you two steps forward. If you are at the bottom of a ladder you move to its top right away. If you are at the mouth of a snake you slide down to its tail right away. You start at 1. The game ends when you reach 9.

(a) Calculate the average number of steps it takes one player to go from start to finish.

(b) Calculate the average number of steps it takes two players to go from start to finish. (Hint: the two players are playing independently, taking turns moving. Hence, we can consider a game where the two players make their moves simultaneously, but using two independent fair coins. The game ends when one of them reaches the finish. Thus, consider a Markov chain with state space being pairs of numbers. The chain starts at $(1,1)$. States (a,b) with a or b being 9 are absorbing. Now, your task is to calculate the number of steps before absorption. The dimension of the matrix is rather large, and so once you set up the problem correctly you will need to use a computer to do the computation.)

(c) John and Mary are playing the game. John starts and then they take turns moving. What is the probability John wins? (Hint: If we instead make the two players move simultaneously, as in part (b), and John's position is represented by the first coordinate, then we are asking for the probability the Markov chain is absorbed at a state of type $(9,b)$. Collapse all these states into one and collapse the remaining ones [of type $(a,9)$ with $a \neq 9$] into one and answer the question.)
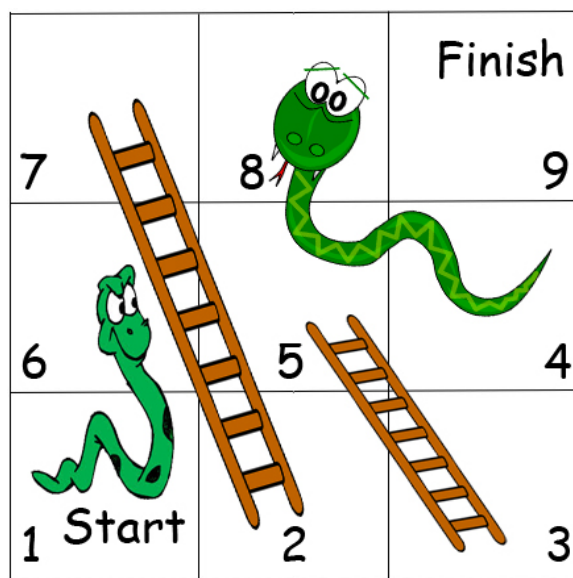
Solution: (a) The only states that are used are $\{1,4,5,7,9\}$. So we have a Markov chain on a state space of size 5. State 9 is absorbing and the other states are transient. The transition matrix is given by

$$
P = \begin{array}{c} \\ 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{array}
\begin{array}{ccccc} 1 & 4 & 5 & 7 & 9 \end{array}
\left[\begin{array}{ccccc}
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1
\end{array} \right].
$$

The transient-to-transient matrix is

$$Q = \begin{array}{c} 1 \\ 4 \\ 5 \\ 7 \end{array}\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \quad \text{and} \quad M = (I-Q)^{-1} = \begin{array}{c} 1 \\ 4 \\ 5 \\ 7 \end{array}\begin{bmatrix} \frac{7}{3} & 1 & \frac{5}{3} & 2 \\ 2 & 2 & 2 & 2 \\ \frac{5}{3} & 1 & \frac{7}{3} & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

(with column labels $1,4,5,7$)

Thus, starting at 1 the palyer will take, on average, $\frac{7}{3}+1+\frac{5}{3}+2 = 7$ steps to complete the game.

(b) Now there are $5 \times 5 = 25$ states of the form (a,b) with a and b in $\{1,4,5,7,9\}$. If any of a or b is a 9, then (a,b) is an absorbing state. The transient-to-transient matrix is given by

$Q =$

	(1,1)	(1,4)	(1,5)	(1,7)	(4,1)	(4,4)	(4,5)	(4,7)	(5,1)	(5,4)	(5,5)	(5,7)	(7,1)	(7,4)	(7,5)	(7,7)
(1,1)	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
(1,4)	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0
(1,5)	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$
(1,7)	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	0
(4,1)	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
(4,4)	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0
(4,5)	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0
(4,7)	0	$\frac{1}{4}$	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0
(5,1)	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0
(5,4)	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0
(5,5)	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
(5,7)	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0
(7,1)	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0
(7,4)	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0
(7,5)	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0
(7,7)	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0

From that we have $M = (I-Q)^{-1} =$

	(1,1)	(1,4)	(1,5)	(1,7)	(4,1)	(4,4)	(4,5)	(4,7)	(5,1)	(5,4)	(5,5)	(5,7)	(7,1)	(7,4)	(7,5)	(7,7)
(1,1)	1.1606	0.1252	0.1036	0.1713	0.1252	0.1092	0.0765	0.1345	0.1036	0.0765	0.3801	0.3665	0.1713	0.1345	0.3665	0.4370
(1,4)	0.1713	1.0928	0.1713	0.2505	0.1345	0.0672	0.1345	0.2185	0.3665	0.1172	0.3665	0.1529	0.4370	0.1008	0.4370	0.2689
(1,5)	0.1036	0.1252	1.1606	0.1713	0.0765	0.1092	0.1252	0.1345	0.3801	0.0765	0.1036	0.3665	0.3665	0.1345	0.1713	0.4370
(1,7)	0.1252	0.0461	0.1252	1.0928	0.1092	0.0252	0.1092	0.0672	0.0765	0.2900	0.0765	0.1172	0.1345	0.3025	0.1345	0.1008
(4,1)	0.1713	0.1345	0.3665	0.4370	1.0928	0.0672	0.1172	0.1008	0.1713	0.1345	0.3665	0.4370	0.2505	0.2185	0.1529	0.2689
(4,4)	0.4370	0.1008	0.4370	0.2689	0.1008	1.1092	0.1008	0.1345	0.4370	0.1008	0.4370	0.2689	0.2689	0.1345	0.2689	0.4370
(4,5)	0.3665	0.1345	0.1713	0.4370	0.1172	0.0672	1.0928	0.1008	0.3665	0.1345	0.1713	0.4370	0.1529	0.2185	0.2505	0.2689
(4,7)	0.1345	0.3025	0.1345	0.1008	0.0672	0.0336	0.0672	1.1092	0.1345	0.3025	0.1345	0.1008	0.2185	0.0504	0.2185	0.1345
(5,1)	0.1036	0.0765	0.3801	0.3665	0.1252	0.1092	0.0765	0.1345	1.1606	0.1252	0.1036	0.1713	0.1713	0.1345	0.3665	0.4370
(5,4)	0.3665	0.1172	0.3665	0.1529	0.1345	0.0672	0.1345	0.2185	0.1713	1.0928	0.1713	0.2505	0.4370	0.1008	0.4370	0.2689
(5,5)	0.3801	0.0765	0.1036	0.3665	0.0765	0.1092	0.1252	0.1345	0.1036	0.1252	1.1606	0.1713	0.3665	0.1345	0.1713	0.4370
(5,7)	0.0765	0.2900	0.0765	0.1172	0.1092	0.0252	0.1092	0.0672	0.1252	0.0461	0.1252	1.0928	0.1345	0.3025	0.1345	0.1008
(7,1)	0.1252	0.1092	0.0765	0.1345	0.0461	0.0252	0.2900	0.3025	0.1252	0.1092	0.0765	0.1345	1.0928	0.0672	0.1172	0.1008
(7,4)	0.1345	0.0672	0.1345	0.2185	0.3025	0.0336	0.3025	0.0504	0.1345	0.0672	0.1345	0.2185	0.1008	1.1092	0.1008	0.1345
(7,5)	0.0765	0.1092	0.1252	0.1345	0.2900	0.0252	0.0461	0.3025	0.0765	0.1092	0.1252	0.1345	0.1172	0.0672	1.0928	0.1008
(7,7)	0.1092	0.0252	0.1092	0.0672	0.0252	0.2773	0.0252	0.0336	0.1092	0.0252	0.1092	0.0672	0.0672	0.0336	0.0672	1.1092

Adding the first row gives that two players will take, on average, 4.042 (simultaneous) steps to end the game. (Do you see why it is natural that the game ends faster with two players than with one?)

(c) We now combine the states of the form $(9, b)$ with $b \in \{1, 4, 5, 7, 9\}$ into one state that we call $W1$ (i.e. player 1 wins) and the states of the form $(a, 9)$ with $a = \{1, 4, 5, 7\}$ into another state we call $W2$ (i.e. player two wins). State $(9, 9)$ belongs to $W1$ because we are assuming player 1 starts and so if both players reach state 9, then player 1 reaches it first and wins. The transient-to-absorbing matrix is then

$$
S = \begin{array}{c} \\ (1,1) \\ (1,4) \\ (1,5) \\ (1,7) \\ (4,1) \\ (4,4) \\ (4,5) \\ (4,7) \\ (5,1) \\ (5,4) \\ (5,5) \\ (5,7) \\ (7,1) \\ (7,4) \\ (7,5) \\ (7,7) \end{array}
\begin{array}{cc} W1 & W2 \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{array}\right] \end{array}
\quad \text{and} \quad
MS = \begin{array}{c} \\ (1,1) \\ (1,4) \\ (1,5) \\ (1,7) \\ (4,1) \\ (4,4) \\ (4,5) \\ (4,7) \\ (5,1) \\ (5,4) \\ (5,5) \\ (5,7) \\ (7,1) \\ (7,4) \\ (7,5) \\ (7,7) \end{array}
\begin{array}{cc} W1 & W2 \\ \left[\begin{array}{cc} 0.5546 & 0.4454 \\ 0.6218 & 0.3782 \\ 0.5546 & 0.4454 \\ 0.3361 & 0.6639 \\ 0.4454 & 0.5546 \\ 0.5546 & 0.4454 \\ 0.4454 & 0.5546 \\ 0.3109 & 0.6891 \\ 0.5546 & 0.4454 \\ 0.6218 & 0.3782 \\ 0.5546 & 0.4454 \\ 0.3361 & 0.6639 \\ 0.6891 & 0.3109 \\ 0.7227 & 0.2773 \\ 0.6891 & 0.3109 \\ 0.6387 & 0.3613 \end{array}\right] \end{array}.
$$

This tells us that if the two players start at 1, then player 1 has a 55.46% chance to win. Also, if, for example, you look at the board and see that player 1 is on 4 and player 2 is on 7 and player 1 is about to play, then player 1 has a 31.09% chance to win.