MATH 5040/6810: Extra exercises

Problem 7

Consider the Markov chain on nonnegative integers that does the following: from 0 it jumps to site $x \geq 0$ with probability $p(x)$ [of course, $p(x) \geq 0$ for all $x \geq 0$ and $\sum_{x=0}^{\infty} p(x) = 1$]. From a site $x > 0$ it goes to $x-1$ with probability 1. In other words, the chain moves to the left until it hits 0, and from 0 it picks a site at random to jump to, then moves to the left again, and so on.

(a) Is this chain irreducible? If not, what are the communicating classes? Is it aperiodic? If not, what are the periods of the different classes?

(b) Starting at state 0, calculate the probability distribution of T_0, the time of first return to 0. What is $E[T_0]$?

(c) Is this chain transient, null recurrent, or positive recurrent? Justify your answer.

(d) Calculate the invariant measure, whenever it exists. What does it mean when the chain does have an invariant measure?

Solution: (a) There is one recurrent communicating class, which is

$$\{x \text{ such that there exists } y \geq x \text{ with } p(y) > 0\}.$$

(Note that 0 is in this class. It is always recurrent, because obviously you will always be able to come back to 0.) The rest of the sites (if there are any) are all transient and are each their own communicating class. Explanation: if there is an integer x for which $p(y) = 0$ for all $y > x$, then sites above this x are inaccessible from 0. So starting at a site above x the chain will move to the left until it hits 0, but then it will never jump back higher than x.

The transient sites (if there are any) in this case do not have a period, because the chain never comes back to them. All recurrent states belong to the same communicating class and thus have the same period, say as 0. One can choose the distribution p to make 0 have any period. For instance, to have period d you can make $p(y)$ positive only on y that are of the form $md - 1$, $m \geq 0$. Then from 0 you can only jump to such a y (that's one step) and then take $md - 1$ steps back to 0, making a total of md steps. I.e. you can only return to 0 in multiples of d steps.

(b) If from 0 the chain jumps to x, then it will take from there x steps to get back to 0. So the time of first return would be $x + 1$. This means that $P\{T_0 = n \mid X_0 = 0\} = p(n-1)$ for all $n \geq 1$. We thus have

$$E[T_0] = \sum_{n=1}^{\infty} np(n-1) = \sum_{k=0}^{\infty}(k+1)p(k) = \sum_{k=0}^{\infty} kp(k) + \sum_{k=0}^{\infty} p(k).$$

The last sum equals one, and so $E[T_0]$ is just one more than the average of the distribution p.

(c) It is enough to study the type of 0. State 0 is always recurrent, because wherever you jump from it you are bound to return. It can be positive or null recurrent depending on whether p has finite or infinite average. That is:

$$0 \text{ is positive recurrent if } \sum_{k=0}^{\infty} kp(k) < \infty \text{ and null recurrent otherwise.}$$

An example where 0 is positive recurrent is to choose $p(k) = \alpha^{k-1}(1-\alpha)$ for some $\alpha \in (0,1)$. Then $E[T_0] = 1/(1-\alpha) + 1 < \infty$. An example where 0 is null recurrent is to choose $p(k) = 1/k - 1/(k+1)$. (Check that $\sum_k p(k) = 1$ and that $\sum_k kp(k) = \infty$.)

Problem 8

Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probability $p(x, x+1) = \frac{x+1}{2x+4}$, $p(x, x-1) = \frac{1}{2}$, and $p(x, x) = 1 - p(x, x+1) - p(x, x-1)$, for $x \geq 1$, $p(0,0) = \frac{3}{4}$, and $p(0,1) = \frac{1}{4}$. Find its invariant measure. Is it transient, null recurrent, or positive recurrent? **Hint:** This is a birth-and-death process.

Solution: This is a birth-and-death process that we studied in class. The general form of this was a process that has state space $0, 1, 2, \dots$, and from $x \geq 1$ the process jumps to $x+1$ with probability a_x, to $x-1$ with probability b_x, and stays at x with probability c_x. From $x = 0$ the process simply does not jump back and so $b_0 = 0$. Then, we found in class that the invariant measure has the formula

$$\pi(x) = \pi(0)\frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x} \quad \text{for all } x \geq 1.$$

In this exercise we have $a_k = \frac{k+1}{2(k+2)}$ and $b_k = \frac{1}{2}$. So

$$a_0 \cdots a_{x-1} = \frac{1}{2(2)} \cdot \frac{2}{2(3)} \cdot \frac{3}{2(4)} \cdot \frac{4}{2(5)} \cdots \frac{x}{2(x+1)} = \frac{1}{2^x(x+1)}$$

and

$$b_1 \cdots b_x = \frac{1}{2^x}.$$

Taking the ratio gives

$$\pi(x) = \frac{\pi(0)}{x+1} \quad \text{for all } x \geq 1.$$

The formula holds also for $x = 0$ because it gives $\pi(0) = \pi(0)$. To determine $\pi(0)$ we use the fact that π needs to add up to 1. But $\sum_{x=0}^{\infty} \frac{1}{x+1} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$ and so there is no way to choose $\pi(0)$ to make the above add up to 1. Hence, there is no invariant measure for this Markov chain. This means it is not positive recurrent. But it could still be null recurrent or transient.

To determine whether the chain is transient we look at the sum

$$\sum_{k=2}^{\infty} \frac{b_1 \cdots b_{k-1}}{a_1 \cdots a_{k-1}} = \sum_{k=2}^{\infty} \frac{\frac{1}{2^{k-1}}}{\frac{4}{2^k(k+1)}} = \sum_{k=2}^{\infty} \frac{\frac{1}{2^{k-1}}}{\frac{4}{2^k(k+1)}} = \sum_{k=2}^{\infty} \frac{k+1}{2} = \infty\,.$$

This says the Markov chain is recurrent. Since we dismissed positive recurrence, the chain is null recurrent, just like the simple symmetric one-dimensional random walk.

Remark: In Problems 5 and 8, when x gets large both Markov chains have transition probabilities that are close to the ones for the simple symmetric random walk. However, note how a small perturbation of the transition probabilities changed the first Markov chain to a transient one while it kept the second Markov chain null recurrent. If you want to practice some more, you can check that swapping the roles of $p(x, x+1)$ and $p(x, x-1)$ in Problem 5 will make the chain positive recurrent. The lesson is that with an infinite state space, the question of transience, positive recurrence, and null recurrence is very subtle and even small perturbations to the transition probabilities could change things drastically.

Problem 9

Suppose we are given numbers $p(x) \in (0,1)$ for all integers $x \geq 0$. Consider the "aging chain" on $\{0, 1, 2, \dots\}$ in which from each $x \geq 0$ the chain moves from x to $x+1$ (gets "older" by 1) with probability $p(x)$ and moves back to 0 ("dies") with probability $1 - p(x)$.

(a) Is this chain irreducible or not?

(b) What conditions on p give transience? **Hint:** Starting at 0, what needs to happen to never return to 0?

(c) What conditions on p give positive recurrence? **Hint:** Calculate the invariant measure.

Solution: (a) The chain is irreducible because for any x and y, if $x < y$ then the chain can just march from x to y and if $y < x$, then then chain can jump from x to 0 and then march to y. So x and y communicate.

(b) Starting at 0, the chain never comes back to 0 only if it keeps marching to the right without ever jumping back to 0. This happens with probability $p(0)p(1)p(2)\cdots$. So the chain is transient if and only if this infinite product is positive and it is recurrent if the infinite product is 0.

(c) The invariant measure satisfies $\pi(x) = p(x-1)\pi(x-1)$ for all $x \geq 1$. Therefore,

$$\pi(x) = p(x-1)p(x-2)\cdots p(0)\pi(0).$$

The chain is thus positive recurrent if

$$\sum_{x=1}^{\infty} p(x-1)p(x-2)\cdots p(0) < \infty$$

and it is not positive recurrent otherwise.