Problem 7

Consider the Markov chain on nonnegative integers that does the following: from 0 it jumps to site $x \geq 0$ with probability $p(x)$ [of course, $p(x) \geq 0$ for all $x \geq 0$ and $\sum_{x=0}^{\infty} p(x) = 1$]. From a site $x > 0$ it goes to $x - 1$ with probability 1. In other words, the chain moves to the left until it hits 0, and from 0 it picks a site at random to jump to, then moves to the left again, and so on.

(a) Is this chain irreducible? If not, what are the communicating classes? Is it aperiodic? If not, what are the periods of the different classes?
(b) Starting at state 0, calculate the probability distribution of T_0, the time of first return to 0. What is $E[T_0]$?
(c) Is this chain transient, null recurrent, or positive recurrent? Justify your answer.
(d) Calculate the invariant measure, whenever it exists. What does it mean when the chain does have an invariant measure?

Problem 8

Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probability $p(x, x+1) = \frac{x+1}{2x+4}$, $p(x, x-1) = \frac{1}{2}$, and $p(x, x) = 1 - p(x, x+1) - p(x, x-1)$, for $x \geq 1$, $p(0, 0) = \frac{3}{4}$, and $p(0, 1) = \frac{1}{4}$. Find its invariant measure. Is it transient, null recurrent, or positive recurrent? **Hint:** This is a birth-and-death process.

Remark: In Problems 5 and 8, when x gets large both Markov chains have transition probabilities that are close to the ones for the simple symmetric random walk. However, note how a small perturbation of the transition probabilities is causing them to behave very differently from the simple symmetric random walk (which we know is null recurrent)!

Problem 9

Suppose we are given numbers $p(x) \in (0, 1)$ for all integers $x \geq 0$. Consider the "aging chain" on $\{0, 1, 2, \dots\}$ in which from each $x \geq 0$ the chain moves from x to $x + 1$ (gets "older" by 1) with probability $p(x)$ and moves back to 0 ("dies") with probability $1 - p(x)$.

(a) Is this chain irreducible or not?

(b) What conditions on p give transience? **Hint:** Starting at 0, what needs to happen to never return to 0?

(c) What conditions on p give positive recurrence? **Hint:** Calculate the invariant measure.