MATH 5040/6810: HOMEWORK 3 (DUE MONDAY, OCTOBER 21)

Problem 1

Prove that the simple symmetric random walk is recurrent in two dimensions. (Hints: (1) To use Pólya's criterion, you need to show that a certain series has an infinite sum. It is enough for that to show that the sum along multiples of four is already infinite. (2) Starting at 0, the walk can return to 0 in $4m$ steps by moving first along one dimension, returning to 0 in $2m$ steps, and then doing the same in the other dimension.)

Problem 2

We know things get worse as dimension increases. So since random walk in one dimension is null recurrent, we expect the two-dimensional case to be no better. Now that you proved the walk in two dimensions is recurrent, prove that it is null recurrent. (Hint: you need to rule out positive recurrence, which means to rule out existence of invariant probability measures.)

Problem 3

Let p and q be two numbers in $[0, 1]$ such that $p + q \le 1$. Consider the Markov chain on nonnegative integers that does the following: from any site $x \ge 0$ it jumps to $x + 1$ with probability p, to $x + 2$ with probability q, and to 0 with probability $1 - p - q$. Is this chain transient? Null recurrent? Positive recurrent? You need to prove whatever you claim.

Problem 4

Consider the Markov chain on $\{0, 1, 2, \dots\}$ in which from each $x \ge 0$ the chain moves from x to $x + 1$ with probability $\frac{1}{x+1}$ and moves back to 0 with probability $\frac{x}{x+1}$.

(a) Is this chain irreducible or not?

(b) Is this chain transient, positive recurrent, null recurrent?

Problem 5

Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probability $p(x, x + 1) = \frac{1}{2}(1 + \frac{1}{x})$ and $p(x, x - 1) = \frac{1}{2}(1 - \frac{1}{x})$ for $x \ge 2$, $p(0, 1) = p(1, 2) = \frac{3}{4}$, and $p(0, 0) = p(1, 0) = \frac{1}{4}$. Show that this chain is transient. (Hint: this is a birth-and-death process.)

Problem 6

Consider the 3×3 board of Snakes and Ladders we studied in class. (See Figure) The rules are: every turn you toss a fair coin. Heads move you one step forward while tails move you two steps forward. If you are at the bottom of a ladder you move to its top right away. If you are at the mouth of a snake you slide down to its tail right away. You start at 1. The game ends when you reach 9.

(a) Calculate the average number of steps it takes one player to go from start to finish.

(b) Calculate the average number of steps it takes two players to go from start to finish. (Hint: the two players are playing independently, taking turns moving. Hence, we can consider a game where the two players make their moves simultaneously, but using two independent fair coins. The game ends when one of them reaches the finish. Thus, consider a Markov chain with state space being pairs of numbers. The chain starts at $(1, 1)$. States (a, b) with a or b being 9 are absorbing. Now, your task is to calculate the number of steps before absorption. The dimension of the matrix is rather large, and so once you set up the problem correctly you will need to use a computer to do the computation.)

(c) John and Mary are playing the game. John starts and then they take turns moving. What is the probability John wins? (Hint: If we instead make the two players move simultaneously, as in part (b), and John's position is represented by the first coordinate, then we are asking for the probability the Markov chain is absorbed at a state of type $(9, b)$. Collapse all these states into one and collapse the remaining ones [of type $(a, 9)$ with $a \neq 9$] into one and answer the question.)